Preprint typeset in JHEP style - HYPER VERSION

From Loop Space Mechanics to Nonabelian Strings

Urs Schreiber

Fachbereich Physik
Universität Duisburg-Essen
Essen, 45117, Germany
E-mail: Urs.Schreiber@uni-essen.de

Abstract: Lifting supersymmetric quantum mechanics to loop space yields the superstring. A particle charged under a fiber bundle thereby turns into a string charged under a 2-bundle, or gerbe. This stringification is nothing but categorification.

We look at supersymmetric quantum mechanics on loop space and demonstrate how deformations here give rise to superstring background fields and boundary states, and, when generalized, to local nonabelian connections on loop space. In order to get a global description of these connections we introduce and study categorified global holonomy in the form of 2-bundles with 2-holonomy. We show how these relate to nonabelian gerbes and go beyond by obtaining global nonabelian surface holonomy, thus providing a class of action functionals for nonabelian strings. The examination of the differential formulation, which is adapted to the study of nonabelian p-form gauge theories, gives rise to generalized nonabelian Deligne hypercohomology. The (possible) relation of this to strings in Kalb-Ramond backgrounds, to M2/M5-brane systems, to spinning strings and to the derived category description of D-branes is discussed. In particular, there is a 2-group related to the String-group which should be the right structure 2-group for the global description of spinning strings.

(July 2005)

Contents

"Stringification is the conversion of
an object to a string [...]."
F. Ribeiro (programmer) [1]

Part I
Overview

In modern formal theoretical physics a certain idea has been found to be very fruitful: **stringification**. Whenever one faces a theory describing particles, one may ask if this theory arises as a limit of a theory where the particles are really one-dimensional strings stretching between their endpoints.

In modern mathematics a certain idea has been found to be very fruitful: **categorification**. Whenever one faces a theory of some algebraic structure describing certain objects, one may ask if this lifts to a structure where objects are replaced by morphisms going between their source and target.

The domains of applicability of these two procedures have a nontrivial intersection where the physics of particles is described by algebra.

This happens in particular when (supersymmetric) quantum mechanics is formulated in terms of spectral triples in Connes' noncommutative spectral geometry (NCG) [2].

Here the configuration space of the particle is encoded in the algebra A of (complex valued) continuous functions over it. This is represented as an operator algebra on a graded Hilbert space $\mathcal{H}$, whose elements describe states of the particle. On this space is defined an odd-graded nilpotent operator D (the "Dirac operator" or "supercharge") which encodes the dynamics of the particle.

This picture of (supersymmetric) quantum mechanics as well its suggestive relation to the RNS superstring, which was pointed out in the second halfs of [3, 4], has been particularly emphasized in [5, 6]. It was noted that superstring dualities find a natural formulation in terms of spectral geometry [7, 8, 9] and further hints for a deeper conceptual rooting of perturbative superstrings in spectral noncommutative geometry were discussed for instance in [10, 11].

Attention to these arguably more conceptual ideas was soon dwarfed by the popularity gained by the noncommutative aspect of NCG that was eventually realized to be ubiquitous in string theory: Open strings in Kalb-Ramond backgrounds were found to give rise to noncommutative field theories [12, 13] and matrix theory formulations of nonperturbative string dynamics [14, 15, 16] resolved smooth spaces by noncommutative matrix algebras. Last not least, string field theory with its noncommutative star product had long been regarded as a manifestation of noncommutative geometry in string theory [17].

On the other hand there is more to noncommutative spectral geometry than just noncommutativity (and in fact a better terminology would be '*not-necessarily commutative* spectral geometry').

For instance the fact that the spectrum of the supercharge in supersymmetric quantum mechanics contains important information about geometric properties of the systems's configuration space (e.g. by way of Morse theory [3] or index theorems) suggests that similarly for instance the spectrum of the string's supercharge should contain interesting information about the configuration space of the string, which is a **loop space** (for closed strings) over target space. And indeed [18] relates this index to elliptic cohomology. This generalized form of cohomology seems alternatively to be obtainable from ordinary 'point geometry' by the method of categorification [19, 20].

It is at this point that one may reasonably suspect that there could be a deeper principle behind lifting spectral geometry and supersymmetric quantum mechanics from points to strings.

1. Preliminaries

The term **nonabelian strings** is supposed to make one think of a generalization of the following situation [21]:

An ordinary particle is a point

$$\bullet$$

which traces out a wordline as time goes by

$$\begin{tikzcd} \bullet \arrow[r, bend left] & \bullet \end{tikzcd} .$$

When the particle is charged, there is a connection in some bundle, which, locally, associates a group element $g \in G$ to any such path

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g"] & \bullet \end{tikzcd} .$$

This happens in such a way that when the particle is transported a little further

$$\begin{tikzcd} \bullet \arrow[r, bend left] & \bullet \arrow[r, bend left] & \bullet \end{tikzcd}$$

the composition of paths corresponds to multiplication of group elements

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g"] & \bullet \arrow[r, bend left, "g'"] & \bullet \end{tikzcd} .$$

It is the associativity of the product in the group that makes this procedure well defined over longer paths

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g"] & \bullet \arrow[r, bend left, "g'"] & \bullet \arrow[r, bend left, "g''"] & \bullet \end{tikzcd}$$

and the existence of inverses which corresponds to the reversal of paths

$$\begin{tikzcd} \bullet & \bullet \arrow[l, bend right, "g^{-1}"'] \end{tikzcd} .$$

The theory of **fiber bundles** with connection tells us how these local consideration fit into a global picture. When the elements g come from a nonabelian group this would be the situation of a **nonabelian particle** which we wish to generalize.

So suppose now that the particle is replaced by a string, which at one moment in time itself already looks like this:

$$\begin{tikzcd} \bullet \arrow[r, bend left] & \bullet \end{tikzcd} ,$$

where the arrow is to remind us of some sort of orientation that we might want to keep track of. Now, as time goes by, this piece of string traces out a worldsheet:

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd} .$$

Is it possible to associate some sort of algebraic object f to this worldsheet

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow f"] & \bullet \end{tikzcd}$$

such that we can make sense of composing pieces of such worldsheet horizontally

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd}$$

and vertically

$$\begin{tikzcd} \bullet \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3ex, "\Downarrow"] \arrow[r, phantom, shift right=3ex, "\Downarrow"] & \bullet \end{tikzcd}$$

and such that multiple compositions like

$$\begin{tikzcd} \bullet \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3ex, "\Downarrow"] \arrow[r, phantom, shift right=3ex, "\Downarrow"] & \bullet \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3ex, "\Downarrow"] \arrow[r, phantom, shift right=3ex, "\Downarrow"] & \bullet \end{tikzcd}$$

are well defined? This is what we would call a theory of **nonabelian strings**. Can we find a global description of this situation that would generalize that of bundles with connection?

We will discuss here that, indeed, one can. This leads to the notion of what we call **2-bundles with 2-connection and 2-holonomy**, which generalize ordinary fiber bundles with connection from the case of points to the case of strings.

1.1 Motivations

There are several motivations for being interested in these kinds of questions:

1.1.1 The Kalb-Ramond Field

First of all there is the well-known abelian situation which we would like to reobtain as a special case of the above idea:

In string theory there is a field, called the Kalb-Ramond field, which locally looks like a 2-form B taking values in the real numbers. Given a piece of worldsheet,Σ, one can (locally) associate the group element

$$\Sigma \mapsto \mathrm{hol}(\Sigma) \equiv \exp\left(i \int_\Sigma B\right) \in U(1)$$

to it. The action functional of the string in this background has the form

$$\exp(iS(\Sigma)) = \exp(iS_{\mathrm{kinetic}}(\Sigma))\, \mathrm{hol}(\Sigma)\ . \tag{1.1}$$

Since $U(1)$ is abelian, the order in which the different $\mathrm{hol}(\Sigma)$ are being multiplied does not matter.

From considerations of "*worldsheet anomalies*" it is well known that the 2-form B globally has to be described by a structure called an *abelian gerbe* [22] and how this nontrivially affects the computation of the *global* definition of hol. A general formalism of nonabelian strings should reproduce all this in appropriate special cases. The formalism which is going to be presented in the following does so. It is however not only more general than that, but also provides a more natural (namely "diagrammatic") language for computing these B-field surface holonomies.

Before proceeding, consider an open string ending on a stack of D-branes (a couple of D-branes on top of each other) in the presence of the Kalb-Ramond field. There is a

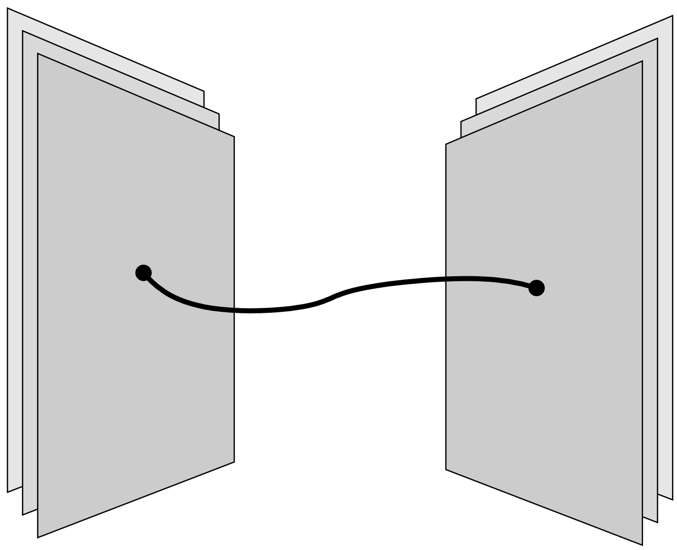


Figure 1: An open string stretching between stacks of D-branes. The bulk of the string couples to an abelian 2-form. The boundary of the string, its endpoints, couple to a nonabelian 1-form.

general argument saying that

- an object with p-dimensional worldvolume coupled to some *abelian* p-form
- can have coupled to its $(p-1)$-dimensional boundary a *nonabelian* $(p-1)$-form.

This argument is sufficient to deduce from the presence of the abelian Kalb-Ramond field alone that the boundary of the open string may couple to a possibly nonabelian 1-form. Hence an open abelian string has nonabelian endpoints. This is precisely the well-known statement that there are possibly nonabelian bundles living on stacks of D-branes.

A nice account of these facts and the following consequence can be found in [23].

1.1.2 Open Membranes on 5-Branes

The above scenario can be "lifted to M-theory". Assuming for simplicity that the stack of D-branes that we started with were D4-branes, this lifts the dimension of everything by one unit: the former open string now becomes an open membrane (the M2-brane) while the 4-branes become 5-branes (the M5-branes). The former nonabelian endpoint of the string on

the stack of branes now becomes an end*string*. This heuristic picture alone already suggests

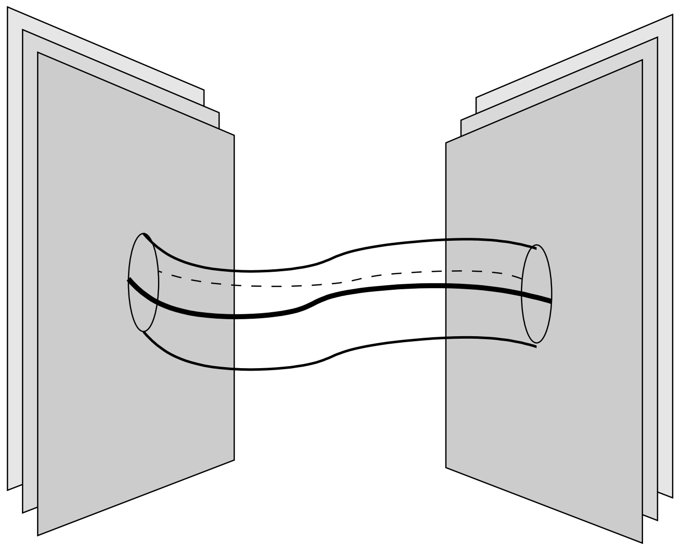


Figure 2: An open membrane stretching between stacks of M-branes. The bulk of the membrane couples to an abelian 3-form. The boundary of the membrane, its end*strings*, are expected to couple to a nonabelian 2-form.

that this endstring is a candidate for a nonabelian string in the above sense. Since the bulk of the membrane couples to the abelian supergravity 3-form, the above argument also leads to the conclusion that the boundary of the membrane should couple to a nonabelian 2-form.

Further arguments for the existence of nonabelian strings on stacks of 5-branes have been given (see §4.1 (p.64)), but still these systems remain notoriously mysterious. A good conceptual understanding of the formal properties of a theory of nonabelian strings should certainly help to shed light on these questions.

We will show how to compute global nonabelian 2-holonomy[1] $\mathrm{hol}(\Sigma)$ for a given surface Σ under certain conditions. This immediately allows to write down candidate action principles for nonabelian strings of the form

$$\exp(iS(\Sigma)) = \exp(iS_{\mathrm{kinetic}}(\Sigma))\,\mathrm{Tr}(\mathrm{hol}(\Sigma)) \ .$$

This is precisely of the same general form as in the abelian case (1.1). The only difference to be taken care of in the nonabelian case is that a suitable operation Tr analogous to the ordinary trace in some representation of the gauge group used in ordinary gauge theory. Whether or not these action principles could pertain to strings on 5-branes is not understood yet, though.

[1]We are using the term "holonomy" where some people might rather say "parallel transport". These people would use "holonomy" for the parallel transport around a closed loop only, while we use the term for parallel transport along any path. When we want to emphasize that we are talking about the holonomy of a closed curve we will speak of *monodromy*.

One important consistency check is related to what is called the N^3-scaling behaviour on theories of 5-branes. It is known that the entropy of ordinary gauge theory asymptotically scales with the square of the rank of the Lie algebra of the gauge group. In the stringy picture this can be thought of as being related to the $\sim N^2$ ways in which the two endpoints of an open string can be attached to N D-branes.

Now, even though the effective field theories on 5-branes are not well understood, there are indirect arguments which indicate that the entropy of these theories should asymptotically scale as N^3, i.e. with the cube of the number of 5-branes involved.

There is a simple heuristic picture making this plausible: The membrane has a certain particularly stable state, called a BPS state, in which it has three disconnected boundary components and hence looks like a pair of pants. Hence in this state there are $\sim N^3$ different possibilities to attach the boundaries of the membrane to one of N 5-branes.

Any formalism of nonabelian strings applicable to M2/M5-brane systems will have to account for this property, somehow. In the formalism developed here there seem to be mechanisms related to that. But this requires further investigation. For more discussion see §4.1.2 (p.66).

1.1.3 Spinning Strings

Even though configurations of M2- and M5-branes are thought to be the fundamental objects in M-theory, these scenarios may look rather exotic. There is however also a much more general way in which nonabelian strings should play a crucial role in string theory.

Whether or not an ordinary particle is charged, it may carry spin. There has to be a spinor bundle with connection which describes how the spin of the particle transforms as it is transported along its worldline.

Superstrings are much like continuous lines of spinning particles. Hence a good global description of spinning strings has to take into account how their spinor degrees of freedom transform as they are parallel transported along their worldsheets. The supercharges of

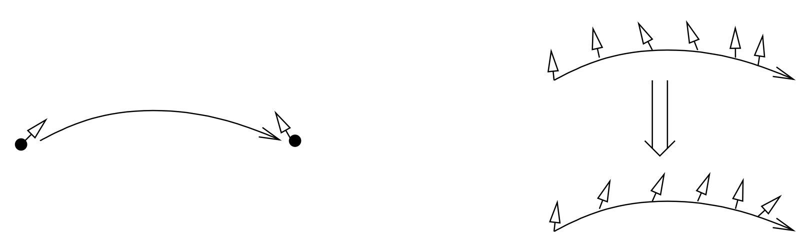


Figure 3: Parallel transport of spinning strings, depicted in the cartoon on the right, is much like the parallel transport of a line of spinning point particles, indicated on the left.

the various flavors of string are generalized Dirac operators on loop space. Given a spinor bundle

$$\begin{array}{c} E \\ \big\downarrow \mathrm{Spin}(n) \\ M \end{array}$$

over spacetime M, one can take loops everywhere and get an $L\mathrm{Spin}(n)$-bundle

$$\begin{array}{c} LE \\ \Big\downarrow L\mathrm{Spin}(n) \\ LM \end{array} .$$

Due to the Virasoro anomaly, this is however not sufficient for the description of superstrings. What is needed is instead a lift of the structure group to a Kac-Moody central extension $\hat{L}\mathrm{Spin}$ of this loop group

$$\begin{array}{c} \hat{L}E \\ \Big\downarrow \hat{L}\mathrm{Spin}(n) \\ LM \end{array} .$$

This is possible only if the first Pontryagin class of the original spin bundle E over M vanishes. In this situation the above can be reformulated by saying that it is possible to lift the structure group of E from $\mathrm{Spin}(n)$ to a group called $\mathrm{String}(n)$.

The topological group $\mathrm{String}(n)$ is defined to be a group which has the same homotopy type as $\mathrm{Spin}(n)$ except that $\pi_3(\mathrm{String}(n))$ vanishes.

These considerations play a role for instance in the computation of the index of the Dirac operator on loop spaces. It is natural to ask if there is a way to capture this somewhat intricate situation with a good concept of nonabelian strings. This indeed turns out to be the case and we will explain how.

More background on spinning strings is recalled in §4.2 (p.68)

1.1.4 Mathematical Motivations

There are several aspects of "higher gauge theory" that are interesting by themselves, for purely mathematical reasons. For quite a while people have already studied aspects of nonabelian surface holonomy for simplicial surfaces in the (comparably simple) case that we call a "trivial 2-bundle" here. For instance topological invariants of knotted surfaces are obtained from counting the number of "flat 2-connections" that one can put on triangulations of these surfaces. There has also been an application of surface holonomy to the four-color theorem [24].

1.1.4.1 Categorification. More generally, the developments presented here fit into a grand framework called **categorification**, which lifts mathematical concepts from sets to 'stringified' sets, called categories. From this point of view the nonabelian strings to be discussed here are but a tiny aspect of an immense structure that mathematicians and physicists (maybe unwittingly) are beginning to explore.

Like a set, a category consists of a collection of **objects**, but unlike a set there are in addition **morphisms** going between pairs of objects in a category. While a map between sets is just a function, a map between categories is called a **functor**. Such a functor takes

morphisms to morphisms, respecting their composition. While the image of two functions can only be equal or not, the image of two functors, being line-like, can be "congruent" (can be translated into each other) without being equal. In this case one says there is a **natural transformation** between these functors.

Given any algebraic structure, we can hence **categorify** it by using the following dictionary [25]:

sets	$\longrightarrow$	**categories**
objects	$\longrightarrow$	morphisms
functions	$\longrightarrow$	functors
equations	$\longrightarrow$	natural transformations .

This is just the first step in an infinite series of categorification steps. Morphisms themselves can be regarded as objects again. The morphisms between these are then **2-morphisms**. We have already encountered this situations in the diagrams at the beginning of §1 (p.9). For instance one can think of a surface Σ (for instance a piece of worldsheet) as a 2-morphism

$$\begin{tikzcd} x \arrow[r, bend left, "\gamma_1"{name=U}] \arrow[r, bend right, "\gamma_2"'{name=D}] & y \arrow[from=U, to=D, Rightarrow, "\Sigma"] \end{tikzcd}$$

between two 1-morphhsism γ_1 and γ_2, which themselves are paths stretching between the objects x and y, which are nothing but points.

More details on concepts from category theory are summarized in §4.3 (p.72).

1.1.5 Category Theoretic Description of Strings

Despite its simplicity, the idea of thinking of a string as a morphism in some category (*cf.* §1.1.4.1 (p.14)), i.e. thinking of a **string as a categorified point particle**, contains in it the seed for essentially all the developments to be discussed here. While this point of view is, among string theorists, rather exotic, there are directions of string research where its ramifications already play a major role.

This is

1. the category theoretic description of conformal and topological 2-dimensional field theories following G. Segal's conception of these issues [26],

2. the description of states of open strings on D-branes in terms of what are called **derived categories**.

There are some obvious vague relations of the approach presented in part III to the first point. The relation to the second point appears to be more subtle but might also be much deeper. We will not try here (nor would there be the space to do so) to elaborate on that in adequate detail. But a brief discussion is given in §4.4 (p.83).

1.2 Outline

The material presented here is mainly a collection of the content of the papers [27, 28, 29], which make up part II, and [30, 31, 32] as well as two papers in preparation [33, 34], constituting part III, equipped with further results and with background information such as to provide a coherent picture of the unifying idea underlying these. Of these, [31] is a collaboration with John Baez, [32] is a collaboration with John Baez, Alissa Crans and Danny Stevenson.

An overview over the material of part II is given in §2 (p.28) and over that of part III is given in §3 (p.33).

The presentation is supposed to be largely self-contained. Throughout part III we make freely use of concepts of (n-)category theory, which the reader can find reviewed in §4.3 (p.72).

An electronic version of this document is available at
http://golem.ph.utexas.edu/string/archives/000578.html.

There are several roads that lead to the considerations presented here. The one we are going to follow starts with supersymmetric quantum mechanics.

- §6 (p.94) [27]

  A system of supersymmetric quantum mechanics (SQM) is specified by giving a C^*-algebra A of observables, called the "position operators", which is represented on a graded Hilbert space $\mathcal{H}$, together with hermitean operators $\left\{D^i\right\}_{i=1,2,\dots,N}$ of odd grade that satisfy the superalgebra

$$\left\{D^i, D^j\right\} = 2\delta^{ij} H$$

  and hence give rise to the **Hamiltonian** H. (We shall be cavalier with technical fine print here, which can be dealt with by the usual standard methods.) For the special case $N = 2$ it is convenient to go over to the nilpotent polar combinations

$$\begin{aligned} \mathbf{d} &\equiv D^1 + iD^2 \\ \mathbf{d}^\dagger &\equiv D^1 - iD^2 \,. \end{aligned}$$

  The notation here is to be suggestive of the archetypical case of SQM, where $\mathcal{H}$ is the Hilbert space of suitable sections of the exterior bundle over some configuration space $\mathcal{M}$ equipped with the Hodge inner product

$$\langle\alpha|\beta\rangle = \int_M \alpha \wedge \star\beta \,,$$

  and where A is the algebra of functions on M and $\mathbf{d}$ the de Rham operator. This can be regared as the point-particle limit of the RR-sector of the RNS superstring.

  We can "turn on background fields" by deforming the supercharges by invertible operators e^W as

$$\begin{aligned} \mathbf{d} &\to e^{-W}\mathbf{d}e^{W} \\ \mathbf{d}^\dagger &\to e^{W^\dagger}\mathbf{d}^\dagger e^{-W^\dagger} \,. \end{aligned}$$

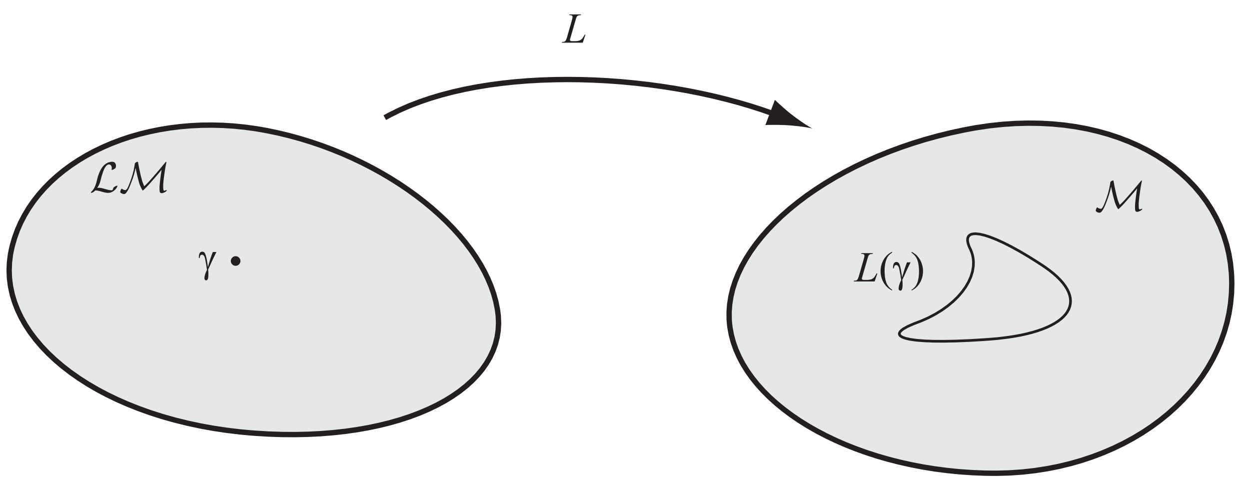


Figure 4: A point γ in loop space $\mathcal{LM}$ maps to a loop $L(\gamma)$ in target space $\mathcal{M}$. Loop space is the bosonic part of the configuration space of the closed string. The full configuration space of the type II superstring is the exterior bundle over loop space.

For instance, when $W \in A$ is just a function this turns on a potential $|\nabla W|^2$. This is because the anticommutator of the deformed supercharges, the deformed Hamiltonian, is the original Hamiltonian, plus a potential term of the form $|\nabla W|^2$, plus fermionic terms (i.e. terms containing differential form creators and annihilators). As another example, when $W = B \in \bigwedge^2 T^*\mathcal{M}$ is the operator of exterior multiplication with a 2-form, the turns on a torsion term $T = \mathbf{d}B$.

Now let $\mathcal{LM}$ be the free loop space over $\mathcal{M}$. We have an exterior derivative $\mathbf{d}$ over loop space. In local coordinates this looks like

$$\mathbf{d} = \int d\sigma \, \mathbf{d}\gamma^\mu(\sigma) \wedge \, \frac{\delta}{\delta\gamma^\mu(\sigma)} \,,$$

where $\mathbf{d}\gamma^\mu(\sigma)\wedge$ is the operator of exterior multiplication by the loop space 1-form $\mathbf{d}\gamma^\mu(\sigma)$, while $\frac{\delta}{\delta\gamma^\mu(\sigma)}$ is the functional derivative.

Hence we can try to lift the above SQM framework from configuration spaces of points to those of strings. When appropriately dealing with subtleties induced by the infinite dimensionality of $\mathcal{LM}$ one finds that $\mathbf{d}$ is related to the fermionic super-Virasoro genrerators G and $\bar{G}$ describing the superstring as

$$\mathbf{d} \propto G + i\bar{G} \,.$$

Similar deformations of $\mathbf{d}$ as for the point particle case can be shown to account for all the massless NS background fields of the string.

In particular, switching on the B-field leads to a deformation

$$\mathbf{d} \to \mathbf{d} + \int \mathrm{ev}^*(B) \wedge + \cdots$$

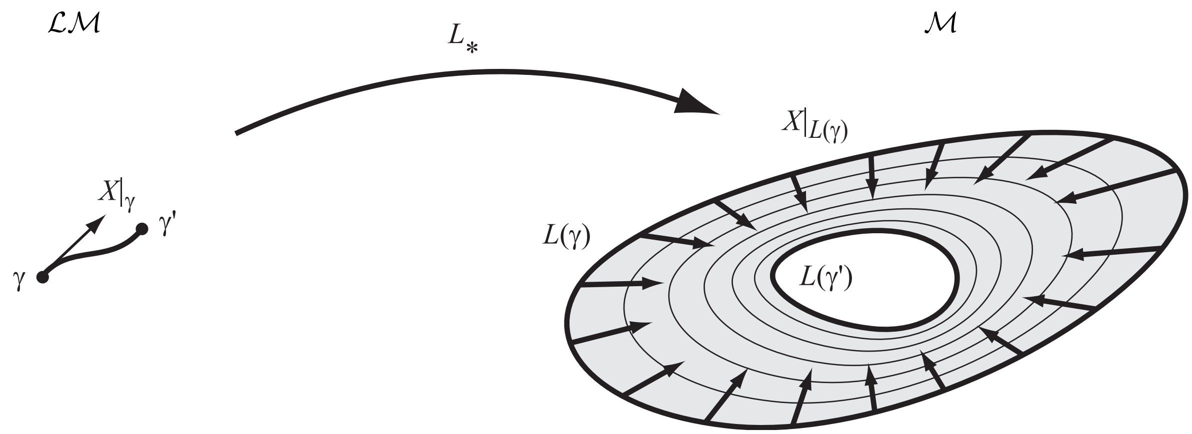


Figure 5: A trajectory in loop space $\mathcal{LM}$ maps to a surface in target space $\mathcal{M}$. String dynamics can be regarded as point dynamics in loop space. Using deformations of string supercharges one obtains local connection 1-forms on loop space. Their line holonomy gives rise to a notion of local surface holonomy hol_i in patches $U_i \subset \mathcal{M}$ of target space.

where the second term denotes a 1-form on loop space obtained by taking the 2-form B on target space, pulling it back to $\mathcal{LM}$ with the **evaluation map**

$$\begin{aligned} \mathrm{ev} : \mathcal{LM} \times S^1 &\to \mathcal{M} \\ (\gamma, \sigma) &\mapsto \gamma(\sigma) \end{aligned}$$

and integrating over S^1. This has the interpretation of an abelian local connection 1-form on loop space. Taking its holonomy over a curve in loop space reproduces the integral of B over the corresponding surface in target space.

It is well known that globally this surface holonomy can be obtained from what is called an abelian gerbe with connection and curving.

There are more interesting deformations that one may consider. Some correspond to gauge transformations of traget space fields, other to superstring dualities.

- §7 (p.132) [28, 35]

  A large class of deformations, those induced by so-called **worldsheet invariants**, has no effect at all on the supersymmetry generators. Still, they act nontrivially on states and indeed can be shown to be related to the **boundary states** describing D-branes with gauge fields.

- §8 (p.161) [30]

  There is a straightforward common generalization of those deformations which induce an abelian 1-form connection on loop space and those that correspond to the

boundary state describing a D-brane with a nonabelian connection 1-form turned on. It leads to a deformation

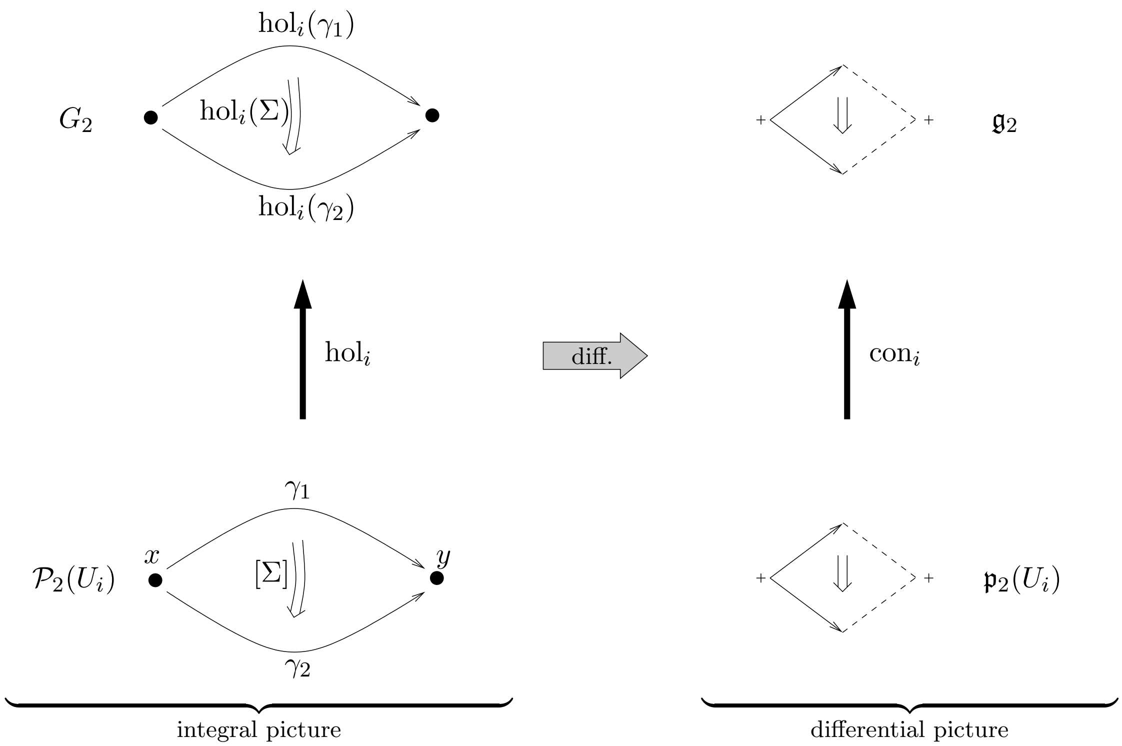


Figure 6: Local 2-holonomy and local 2-connection is the higher dimensional generalization ("categorification") of local holonomy and local connection. Local 2-holonomy is a 2-functor hol_i that maps surface elements in a 2-path 2-groupoid $\mathcal{P}_2(U_i)$ to elements of a categorified Lie group (Lie 2-group) G_2. Differentially, this comes from a 2-connection con_i which can be realized as a 2-functor from the 2-path 2-algebroid $\mathfrak{p}_2(U_i)$ to the Lie 2-algebra $\mathfrak{g}_2$. Such a local 2-connection is specified by a 1-form A and a 2-form B, taking values in $\mathfrak{g}_2$.

$$\mathbf{d} \to \mathbf{d} + \int W_A(\mathrm{ev}^*(B)) \wedge + \cdots ,$$

where now $B \in \Omega^2(M, \mathfrak{h})$ takes values in a possibly nonabelian Lie algebra $\mathfrak{h}$ and where W_A denotes the parallel transport of B to the origin of the loop by means of a connection 1-form $A \in \Omega^1(M, \mathfrak{g})$ taking values in a Lie algebra $\mathfrak{g}$ which acts on $\mathfrak{h}$. In order for this to be meaningful, $\mathfrak{g}$ and $\mathfrak{h}$ have to form what is called a **differential crossed module** $(\mathfrak{g}, \mathfrak{h}, d\alpha, dt)$ where α and t are Lie algebra homomorphisms

$$\begin{aligned} d\alpha &: \mathfrak{g} \to \mathrm{Der}(\mathfrak{h}) \\ dt &: \mathfrak{h} \to g \end{aligned}$$

satisfying a certain compatibility condition.

The physics described by this deformation can no longer be that of D-branes. The connection 1-form on loop space is now nonabelian and hence integrating it over a

curve in loop space yields a nonabelian group element associated to the corresponding worldsheet in target space. This suggests that it describes nonabelian strings. For this to make good sense, a global description of these nonabelian connections is necessary. It turns out that the above connection 1-forms are precisely those that appear in a *categorified* version (*cf.* §1.1.4.1 (p.14)) of ordinary fiber bundles, called 2-bundles, which should hence provide precisely this global description.

The relation of the above loop space formalism to categorification was suspected when it was found that for the above nonabelian connection 1-form to yield a reparameterization invariant surface holonomy in target space and to behave sensibly under gauge transformations, the following relation between the 2-form B and the field strength F_A of the 1-form needs to hold:

$$dt(B) + F_A = 0 \,.$$

It turned out that this relation was encountered before, in the study of categorified lattice gauge theory by Girelli and Pfeiffer.

We will discuss this condition in detail in the main text. The reader is reminded that what are, conventionally, called B and A now can no longer be the fields of the same name in the context of abelian strings on stacks of D-branes.

- §10 (p.200) [32]

  In categorifying gauge theory, one of the crucial steps is to find a categorification of the concept of gauge group. The result of applying the dictionary in §1.1.4.1 (p.14) to the definition of an ordinary group is called a **2-group** or "gr-category".

  In ordinary gauge theory one associates group elements to pieces of worldlines. In categorified gauge theory one instead associates morphisms of a 2-group to pieces of worldsheet.

  All perturbative superstrings, regardless of the backgrounds that they propagate in, carry spin degrees of freedom. Hence there should be a 2-group related to the Spin-group which describes the parallel transport of spinning strings. It turns out that a known Lie 2-algebra, which is called $\mathfrak{spin}_1$ and which in a subtle way is what is called "non-strict", is categorically equivalent to the Lie 2-algebra of a Lie 2-group called $\mathcal{P}_1\mathrm{Spin}(n)$, which has the right properties to do just that.

- §11 (p.242) [31]

  Ordinary gauge theory requires the notion of a principal fiber bundle. This is a total space E together with a projection $E \to M$ of this space onto spacetime M, such that over contractible patches $U_i \subset M$ of spacetime the total space looks like $E|_{U_i} \simeq U_i \times G$, i.e. like spacetime with a copy of the "gauge group" G attached to each point.

  When the categorification dictionary displayed in §1.1.4.1 (p.14) is applied to this structure, one ends up with a category E, a category M and a functor $E \to M$, such

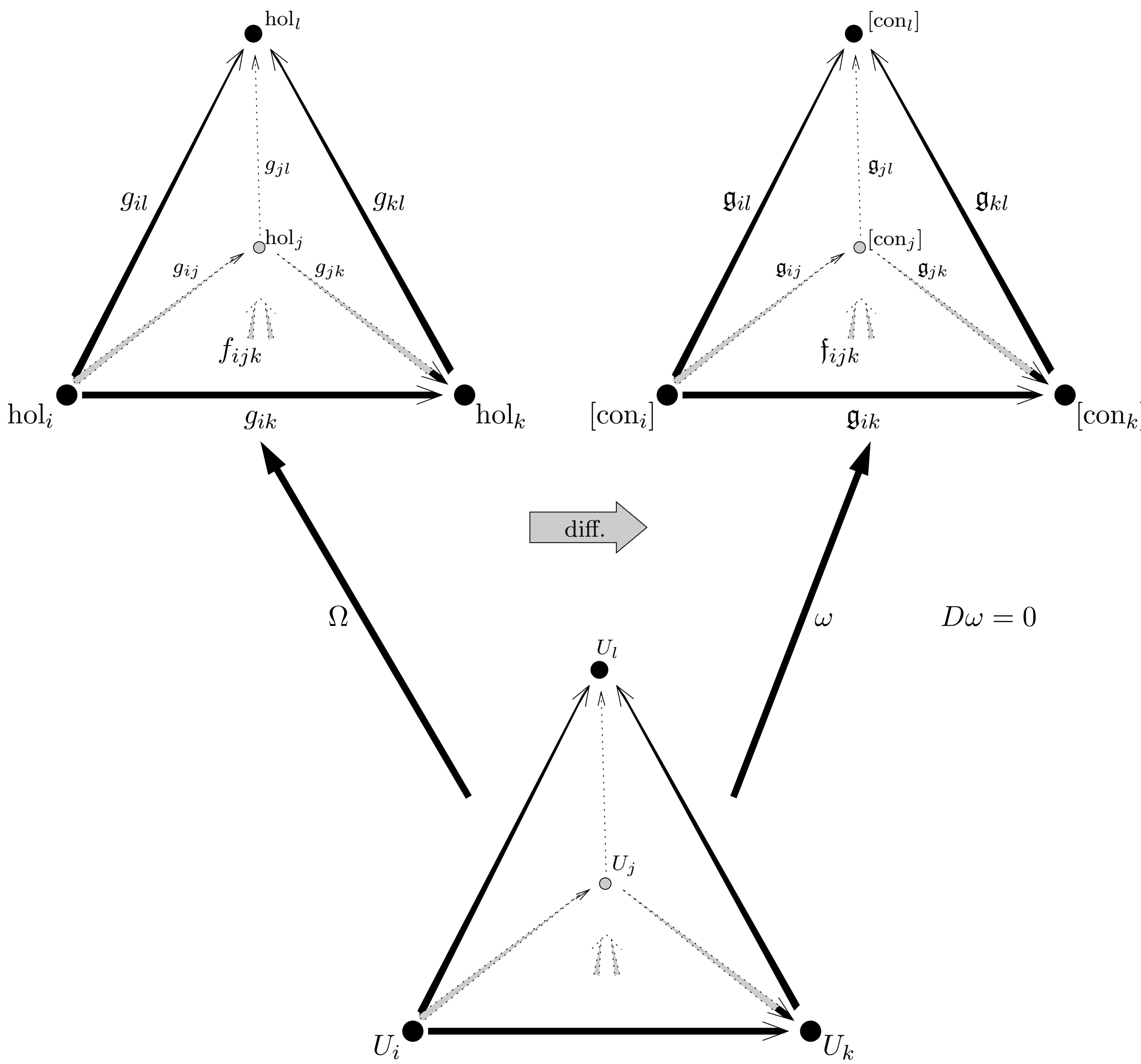


Figure 7: A **2-Bundle with 2-Holonomy** over an ordinary base space B is, when locally trivialized with respect to a good covering $\mathcal{U} = \bigsqcup_{i\in I} U_i$ of B, an assignment Ω of a) local 2-holonomy 2-functors hol_i to patches U_i, b) of pseudo-natural transformations $\mathrm{hol}_i \xrightarrow{g_{ij}} \mathrm{hol}_j$ to double overlaps U_{ij}, and c) of modifications $g_{ik} \xrightarrow{f_{ijk}} g_{ij} \circ g_{jk}$ of such transformations to triple overlaps U_{ijk}, such that the tetrahedron on the left 2-commutes. (There is a 2-morphisms in every face of this tetrahedron, but for convenience only one of them is displayed.) Differentially, this is an assignment ω of 2-connections con_i to patches U_i and of 1-morphisms $\mathfrak{g}_{ij}$ and 2-morphisms $\mathfrak{f}_{ijk}$ between these to double and triple overlaps, respectively, such that the tetrahedron on the right 2-commutes. This is equivalent to saying that ω is a cocycle with respect to a generalized (nonabelian) Deligne coboundary operator D. Gauge transformations correspond to homotopies of the map Ω, which in the differential picture comes from shifts by D-exact elements: $\omega \to \omega + D\lambda$.

that E locally looks like $U_i \times G$, where G is now a 2-group. This is called a **2-bundle** [36].

In order to find how a 2-bundle describes nonabelian strings, one needs to furthermore categorify the notion of *connection of a bundle* such that it admits a categorification of the notion of *holonomy of a connection*.

One nice way to describe the concept of an ordinary connection on an ordinary principal bundle uses the idea of a functor. One can regard the set of *paths* (worldlines) in spacetime as a category whose objects are all the points of spacetime and whose morphisms are all paths between pairs of these points. One can also regard an ordinary (gauge) group as a category with a single object and one morphism for every group element. An ordinary connection is then nothing but a functor hol_i from the category of paths in contractible patches U_i of spacetime to the gauge group (*cf.* §4.3.1.2 (p.76)). This is just the formal version of the familiar statement that a connection allows to do "parallel transport" along any given path.

On double overlaps $U_{ij} = U_i \cap U_j$ of two contractible patches U_i and U_j the parallel tranports induced by hol_i and hol_j are related by a *gauge transformation* g_{ij}. In terms of functors this is nothing but a natural transformation (*cf.* §4.3.1.3 (p.77)) between hol_i and hol_j. On triple overlaps these transformations have to satisfy the familar consistency identity $g_{ij} \circ g_{jk} = g_{ik}$.

Now, a 2-connection with 2-holonomy on a 2-bundle is the categorification of this situation. So it is locally on U_i a 2-functor (*cf.* §4.3.2.1 (p.81)) hol_i , which assigns elements of a 2-group G_2 to surface elements in U_i. On double overlaps , hol_i and hol_j are related by a pseudo-natural transformation g_{ij} of 2-functors. On a triple intersection the natural transformations g_{ik} and $g_{ij} \circ g_{jk}$ are themselves related by a morphism f_{ijk} between natural transformations, called a modification. These f_{ijk} finally satisfy a certain consistency condition on quadruple overlaps.

It turns out that in the case that the gauge 2-group G_2 has a property called "strictness", a 2-connection with 2-holonomy locally comes from a 2-functor hol_i which itself is determined precisely by the connection 1-form

$$\mathcal{A}(\gamma) = \int\limits_\gamma W_A(\mathrm{ev}^*(B))$$

on the space of paths γ that we encountered before in the context of deformations of SQM on loop space. It turns out that the condition

$$F_A + dt(B) = 0$$

arises as consequence of *functoriality*, i.e. of the fact that functors respect composition of morphisms.

- <u>§12 (p.285) [34]</u> With the concept of 2-connection with 2-holonomy in a principal 2-bundle thus available, it is now possible to compute the surface holonomy of any given surface with respect to this 2-connection.

It turns out that it is possible to glue the local 2-holonomy 2-functors hol_i on every patch U_i into a global 2-holonomy 2-functor by using 2-group elements that enter the definition of the transition morphisms g_{ij} and f_{ijk}. This is indicated in figure 8.

One can give a more intrinsic description of this situation, one that does not make recourse to a choice of good covering, in terms of a single global 2-functor

$$\mathrm{hol}\colon \mathcal{P}_2(M) \to G_2\!-\!2\mathrm{Tor}$$

that maps 2-paths in all of base space not to the structure 2-group G_2, but to the category of G_2-2-*torsors*. For an ordinary group G, a (left) G-torsor is a space which has a free and transitive (left) G-action, i.e. which is a left G-space, and which furthermore is isomorphic to G as a G-space. But not necessarily canonically isomorphic. The fiber of an ordinary principal G-bundle is a G-torsor. It is well known how an ordinary principal bundle with connection is given by a global 1-holonomy 1-functor

$$\mathrm{hol}\colon \mathcal{P}_1(M) \to G\text{–}\mathrm{Tor}$$

from paths in base space to the category G–Tor of G-torsors. This category has G-torsors as objects and G-torsor morphisms as morphism. This are maps between torsors that are compatible with the left G-action.

More precisely, given a G-bundle E, we have the smooth category $\mathrm{Trans}_1(E)$ whose objects are the fibers E_x of E, regarded as G-torsors, and whose morphisms are the G-torsor morphisms between these fibers. When we forget about the smooth structure of $\mathrm{Trans}_1(E)$ we can regard it as a subcategory of G–Tor and our global 1-holonomy 1-functor looks like

$$\mathrm{hol}\colon \mathcal{P}_1(M) \to \mathrm{Trans}_1(E)\ .$$

A 2-torsor is the obvious categorification of the concept of a torsor. There is a 2-category G_2–2Tor of G_2-2-torsors. Similarly, when E is a principal G_2 2-bundle with connection and holonomy it is specified by a global 2-holonomy 2-functor

$$\mathrm{hol}\colon \mathcal{P}_2(M) \to \mathrm{Trans}_2(E)\ ,$$

where now $\mathrm{Trans}_2(E)$ is the 2-category whose objects are the fibers E_x of the G_2-2-bundle E, regarded as G_2-2-torsors.

This is the most elegant description of principal 2-bundles with 2-connection and 2-holonomy that we are discussing here.

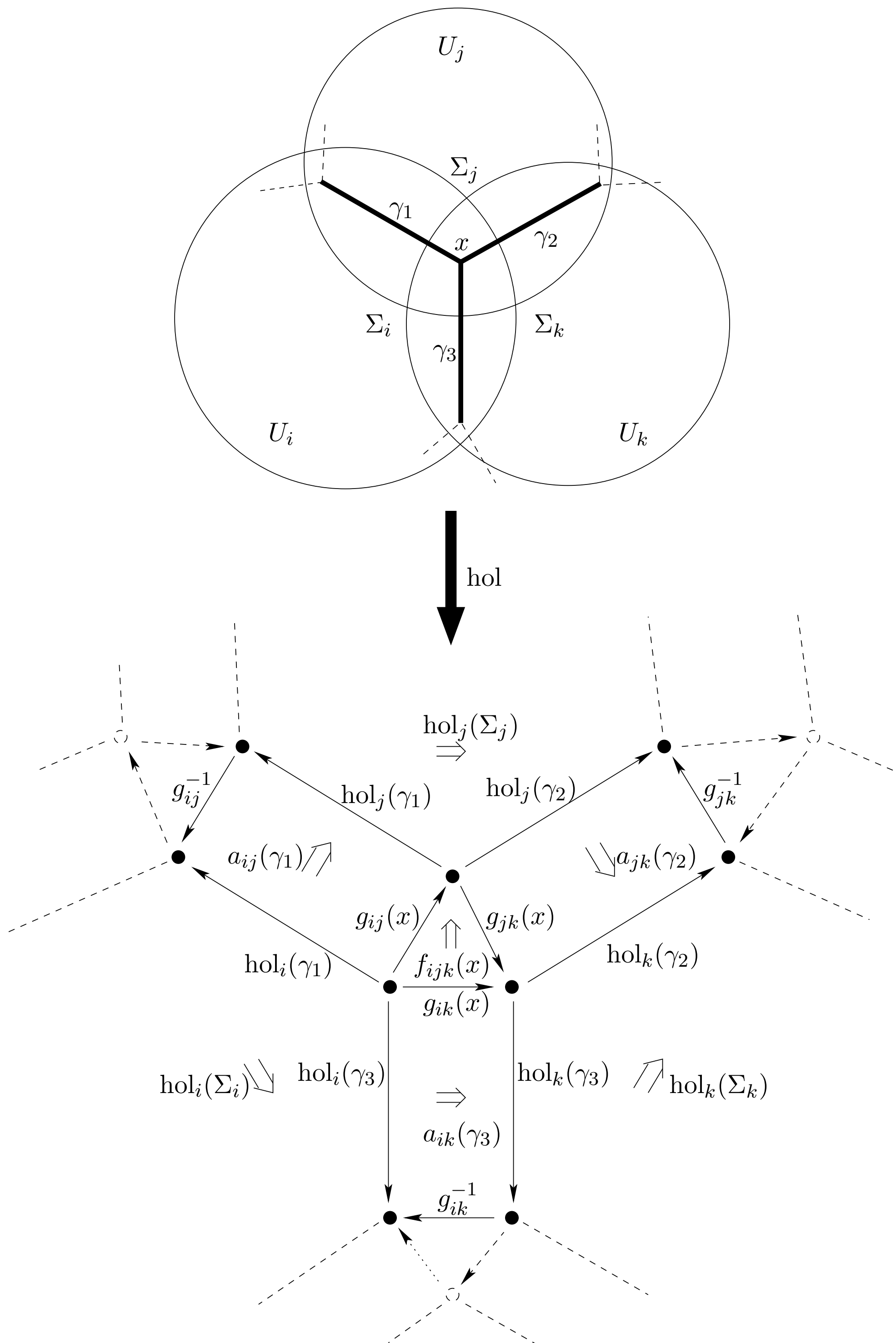


Figure 8: Global surface holonomy of a surface Σ is obtained from the local 2-holonomy 2-functors hol_i by suitably gluing them together. First triangulate Σ such that each face Σ_i sits in a single patch U_i. Then assign the local 2-holonomy $\mathrm{hol}_i(\Sigma_i)$ to these faces. Certain 2-group elements $a_{ij}(\gamma)$ (coming from the transition on double overlaps) are assigned to edges γ and 2-group elements $f_{ijk}(x)$ (coming from the transition on triple overlaps) are assigned to vertices x of the triangulation. The global 2-holonomy is then the well-defined composition of all these 2-group elements. In a special simple case this reproduces the well-knonw formula for surface holonomy in abelian gerbes with connection and curving.

Recalling that the exponentiated action functional for a nonabelian particle is the kinetic term times the holonomy along the worldline, we can thus write down exponentiated action functionals for *nonabelian strings* by multiplying the usual kinetic term with the above notion of surface holonomy over the worldsheet of the string.

$$\exp(iS(\Sigma)) = \exp(iS_{\text{kinetic}}(\Sigma)) \; \mathrm{Tr}(\mathrm{hol}(\Sigma)) \; ,$$

where Tr is a suitable operation that maps morphisms of a 2-group to complex numbers in a gauge invariant way.

- §13 (p.335) [33]

  Instead of working with p-holonomy functors hol_i that associate p-group elements to p-dimensional volumes, one can go to the differential description of these. This leads to functors that associate Lie p-algebra morphisms to p-forms and provides a complementary perspective on the above issues, which for instance provides a powerful formalism for writing down action principles for higher p-forms such as B [37]. It also provides a nonabelian generalization of Deligne hypercohomology, which allows to conveniently handle p-bundles with p-connection and p-holonomy using cohomological methods.

1.3 Acknowledgments

This research first and foremost owes its existence and nature to the beneficial working environment provided by Prof. R. Graham, who offered the liberty to do autonomous research together with his valuable guidance and advice. The idea of applying SQM deformation methods to supergravity systems goes back to him. This idea I had ample chance to investigate in my master thesis, and building on that I could apply these deformations to the string's worldsheet supergravity (§II), which is the starting point for most of the considerations reported here.

Sections §9 (p.182), §10 (p.200) and §11 (p.242) are due to a very fruitful and most inspiring collaboration with John Baez. The integration of the loop space formalism from the first half of this work (§II) into a theory of 2-connections on 2-bundles is ongoing joint work with him. It is the timely appearance of Toby Bartels' definition of 2-bundles in [36] which was crucial for making our collaboration possible in the first place.

Even though I am solely responsible for a couple of further developments on 2-bundles which are presented here, like the discussion of global 2-holonomy and of 2-gauge transformations in principal 2-bundles (§12), of strict 3-groups and 3-bundles (§10.7 and §12.3 (p.313)) and of vector 2-bundles (§4.4.2), all of these have benefited from discussion with John Baez and would hardly have been conceived without his influence on my thinking.

The research which lead to the results concerning the 2-group $\mathcal{P}_k G$ in §10 was to some extent motivated by a comment by Edward Witten regarding a possible relation of elliptic cohomology to our 2-connections, as well as by the announcement of a talk by André Henriques on a relation between the Lie 2-algebra $\mathfrak{g}_k$ (§10.2.3) and the group $\mathrm{Spin}(n)$. The results presented in this section are due to joint work with John Baez, Alissa Crans and Danny Stevenson and taken from our paper [32]. This work started while I had the chance to visit John Baez's group at UC Riverside in February 2005. I am most grateful for this kind invitation and for the intensive discussions we had there, many aspects of which have found their way into the presentation given here. I also learned a lot from many exchanges of ideas with Danny Stevenson since then, who helped me open the door to the world of gerbes and bundle gerbes.

I am much obliged to several other people who have shown interest in results of my work by inviting me, giving me the opportunity to talk about my ideas and providing valuable feedback. I had the opportunity to visit (in this order) Ioannis Giannakis at Rockefeller University in New York, talking about deformations of conformal field theories; Hermann Nicolai at the Albert-Einstein-Institute in Golm, and Rainald Flume at the University of Bonn, who were interested in my papers on string quantization related to Pohlmeyer and DDF invariants; Paolo Aschieri and Branislav Jurčo, who I met at University of Torino where we talked about nonabelian gerbes and 2-bundles; Christoph Schweigert at University of Hamburg, who listened to what I had to say about 2-bundles and 2-connections, Branislav Jurčo once again who also invited me to University of Munich for further discussion; and Thomas Strobl at University of Jena.

Apart from those people that I had the chance to meet in person, there are several with whom I had helpful discussion by electronic means on various topics related to my

work. In this context I want to thank Jacques Distler for setting up the weblog *The String Coffee Table* [38] for this purpose, and for equipping it with the high standard of technology for math on the web that it has. I am indebted to all participants of discussions on this weblog.

For the first part of my research this includes most notably Eric Forgy. Our intensive and very enjoyable collaboration on discrete differential geometry by means of deformed spectral triples has lead to the preprint [39], several aspects of which reappear, in one guise or another, in the discussion of CFT deformations in §6. I am deeply indebted to Eric for taking genuine interest in my ideas and for all our very constructive discussions. Last not least, I thank him for providing figures 4, 5, 9 and 10. I could hardly ever have created these myself.

When I thought about the issues that are now discussed in §4.4, it was Aaron Bergman who provided a lot of help with pointers to the literature on various aspects of derived categories in string theory and discussion of technical details, as well as on the underlying principle that one might or might not suspect here.

I also benefited from comments by Robert Helling, Andrew Neitzke and Luboš Motl on N^3-scaling behaviour in 5-brane theories.

Other participants of discussions about categorified gauge theory that I am grateful for are Orlando Alvarez, Jens Fjelstad, Amitabha Lahiri, and Thomas Larsson. With Jens Fjelstad I had some interesting personal discussion about the nature of 2-curvature in 2-bundles, which has become part of the exposition in §11.6 (p.281).

I have received valuable comments while this document was being proofread from John Baez, Robert Helling and Branislav Jurčo. Of course all remaining imperfections are mine. Many thanks to Axel Pelster for his help with formatting issues.

Finally, I heartily thank Philip Kuhn for his concern about my water balance and for the most kind continuous supply with herbal tea that he provided. Without him this research might have decayed to dust before it was even finished.

This work was supported by SFB/TR 12.

2. SQM on Loop Space

Start by considering ordinary **supersymmetric quantum mechanics**, consisting of a graded Hilbert space $\mathcal{H}$ on which an algebra A of 'position operators' and $N = 1, 2, \ldots$ odd-graded, self-adjoint '**Dirac operators**' or '**supercharges**' D^i are represented, which determine the Hamiltonian H by the relation

$$\left\{D^i, D^j\right\} = 2\delta_{ij} H \,,$$

called the $D = 1$, $N = 1, 2, \ldots$ (Poincaré) **supersymmetry algebra**.

The triple $\left\{A, \mathcal{H}, D^i\right\}$ is alternatively known as a **spectral triple** and can be seen as an algebraic description of the geometry of configuration space.

For $N = 2$, in particular, the nilpotent linear combinations $\mathbf{d} \propto D^1 + iD^2$ and $\mathbf{d}^\dagger \propto D^1 - iD^2$ are of interest. Given any 1-parameter family $\exp(W(t))$ of invertible operators on $\mathcal{H}$, the deformation

$$\begin{aligned} \mathbf{d} &\to e^{-W} \circ \mathbf{d} \circ e^{W} \\ \mathbf{d}^\dagger &\to e^{W^\dagger} \circ \mathbf{d}^\dagger \circ e^{-W^\dagger} \end{aligned}$$

preserves the superalgebra and hence defines a new system of supersymmetric quantum mechanics.

Note that this is a global similarity transformation which leaves the physics unaffected *only if* the deformation operator W is anti-hermitean, in which case the above describes gauge transformations.

The standard example of supersymmetric quantum mechanics is the case where $\mathbf{d}$ is the exterior derivative on some manifold M, A is the algebra of continuous (real- or complex-valued) functions on M and $\mathcal{H}$ is the Hilbert space of suitably well-behaved sections of the exterior bundle over M, equipped with the Hodge scalar product $\langle\alpha|\beta\rangle = \int \alpha \wedge \star\beta$. Choosing the deformation W to be in A introduces a scalar potential $|\nabla W|^2$ (a 'background field'!) into the Hamiltonian H, which is famously related to the Morse theory of M. This setup (for $W = 0$) can be thought of as giving the point-particle limit of the R-R sector of the RNS superstring.

This already suggests that there is nothing more natural than replacing M with LM, the **free loop space** over M, $\mathbf{d}$ with the exterior derivative on LM, and so on. In other words this amounts to switching from the spectral triple for the configuration space M of a particle to that of the configuration space LM of a closed string.

When we think of loop space as locally coordinatized by the set $\{\gamma^\mu(\sigma)\} \equiv \left\{\gamma^{(\mu,\sigma)}\right\}$ of coordinates, where $\gamma : [0, 2\pi] \to M$ is a parameterized loop, then for instance the exterior derivative locally reads

$$\mathbf{d} = \int d\sigma \; \mathbf{d}\gamma^\mu(\sigma) \wedge \frac{\delta}{\delta\gamma^\mu(\sigma)} \,,$$

where $\frac{\delta}{\delta\gamma^\mu(\sigma)}$ is the functional derivative.

Taking care of issues related to the infinite-dimensionality of LM one finds that the **super-Virasoro** generators represented on the Hilbert space of the closed superstring

Figure 9: *A vector on loop space* does not necessarily induce a vector field on a loop in target space

(i.e. the left- and right-moving parts T and $\bar{T}$ of the worldsheet energy-momentum tensor as well as the corresponding supercurrent with components G and $\bar{G}$) indeed provide a supersymmetric quantum mechanics on loop space in the above sense. For instance for a purely gravitational background the polar combination

$$G_0 + i\bar{G}_0 \propto \mathbf{d}_K$$

is proportional to the exterior derivate $\mathbf{d}$ on loop space summed with the operator K of inner multiplication with the generator of reparameterizations of loops.

2.1 Deformations and Background Fields

One may hence ask what deformation operators W do to this system, i.e. what dynamics the deformed operator

$$\begin{aligned} \mathbf{d}_K \to \quad & e^{-W}\, \mathbf{d}_K\, e^{W} \\ & \propto e^{-W}\, (G_0 + i\bar{G}_0)\, e^{W} \end{aligned}$$

describe.

It turns out that all massless NS-NS **background fields** of the superstring can be encoded in a suitable deformation W of the loop space spectral triple.

For instance when choosing

$$W_B(\gamma) \propto \int_\gamma d\sigma\ B_{\mu\nu}(\gamma(\sigma))\, \mathbf{d}\gamma^\mu(\sigma) \wedge \mathbf{d}\gamma^\nu(\sigma) \wedge$$

for a (possibly only locally defined) 2-form B on target space and with γ a point in loop space, the deformed super-Virasoro operators are those that otherwise follow from a canonical analysis of the supersymmetric σ-model for the Kalb-Ramond background described

Figure 10: *The reparameterization Killing vector* on parameterized loop space always exists

by B, i.e. from the supersymmetric σ-model with action

$$S = \frac{T}{2} \int d^2\xi d^2\theta \ (G_{\mu\nu} + B_{\mu\nu})\, D_+ \mathbf{X}^\mu D_- \mathbf{X}^\nu \,,$$

where θ is a Grassmann variable and $\mathbf{X}$ the worldsheet superfield.

In particular, the deformed $\mathbf{d}_K$ reads

$$e^{-W_B} \mathbf{d}_K e^{W_B} = \mathbf{d}_K - iT \int_\gamma d\sigma \ B_{\mu\nu} \gamma'^\nu \mathbf{d}\gamma^\mu + \frac{1}{6} \int_\gamma d\sigma \ (\mathbf{d}B)_{\alpha\beta\rho} d\gamma^\mu \wedge d\gamma^\nu \wedge d\gamma^\rho \,.$$

The first new term on the right is the B-field pulled back to and integrated over the given loop. The resulting loop space 1-form has the interpretation of a local **connection 1-form on loop space**. The other is the field strength of B, which is interpretable as a torsion term. The proper global framework for these quantities is well known to be that of **abelian gerbes** with connection and curving, which we will reproduce in part III as a special case of 2-bundles with 2-connection.

By expanding the above deformations to first order in the background fields it is found that they produce the well-known so-called **canonical deformations** of 2D conformal field theories.

Moreover, deformation operators W which are anti-hermitean should give rise to gauge transformations of our system, since for them (and only for them) the above deformation degenerates to a global similarity transformation. Indeed, such operators can be shown to describe **gauge transformations** of background fields as well as **T-duality** operations on the background.

2.2 Worldsheet Invariants and Boundary States

Another special role is played by deformations e^W which commute with $\mathbf{d}_K$ and all of its modes.

Among them are the **worldsheet invariants**, namely those observables which commute with *all* the super-Virasoro generators. Traditionally these are known in their incarnation as **DDF invariants**. These can be shown to be essentially equivalent to the set of what are called (supersymmetric) **Pohlmeyer invariants**.

Deforming by such operators evidently does not lead to any effective deformations at all when conjugating $\mathbf{d}_K$. However, they are still of interest as deformation operators (apart from their main interest as invariant observables of the string):

It can be seen that the constant 0-form $\mathbf{1}$ on loop space is nothing but the **boundary state** describing the bare, space-filling $D9$-brane. It turns out that $\mathbf{d}_K$-closed deformations W give rise to **boundary state deformations**

$$\mathbf{1} \to e^W \mathbf{1} \,.$$

One finds that the deformation

$$\mathrm{Tr}\,\mathrm{P}\exp\left(\int_0^{2\pi} d\sigma\ \left(iA_\mu \gamma'^\mu + \frac{1}{2T}(F_A)_{\mu\nu}\mathbf{d}\gamma^\mu \wedge \mathbf{d}\gamma^\nu \wedge\right)\right)\mathbf{1}$$

which assigns to each element of loop space its supersymmetric Wilson line with respect to some gauge field A, corresponds to the boundary state obtained by turning on that gauge field on a stack of D-branes. The (supersymmetric) Pohlmeyer invariants, which themselves have the rough form of Wilson lines, give rise to such boundary states when applied to the 0-form $\mathbf{1}$.

All this holds classically in general, while at the quantum level one encounters the usual divergences which should vanish (as has been checked to low order) when the background fields satisfy their equations of motion.

This way there is a nice correspondence between algebraic deformations of spectral triples on loop space and various aspects of known string physics.

2.3 Local Connections on Loop Space from Worldsheet Deformations

From the loop space perspective there is a natural generalization of the above inhomogenous differential form on loop space, namely

$$(e^W)_{A,B} \equiv \mathrm{Tr}\,\mathrm{P}\exp\left(\int_0^{2\pi} d\sigma\ \left(iA_\mu \gamma'^\mu + \frac{1}{2T}(F_A + B)_{\mu\nu}\mathbf{d}\gamma^\mu \wedge \mathbf{d}\gamma^\nu \wedge\right)\right)\ ,$$

where B is a Lie-algebra-valued 2-form. For nonvanishing B this no longer commutes with $\mathbf{d}_K$. Instead one finds that

$$(e^W)^{-1}_{A,B}(\mathbf{d}_K (e^W)_{A,B}) = iT \oint_A (B) + (\text{terms of grade} > 1)$$

where the term on the right denotes the loop space 1-form obtained by pulling B back to the given loop and intergrating it over that loop while continuously parallel transporting it to

the basepoint using the algebra-valued 1-form A. This can be interpreted as a **nonabelian connection 1-form** on loop space.

Hence this cannot describe the boundary state for a fundamental string on a D-brane anymore. There is also no nonabelian 2-form field living on D-branes.

There are, however, nonabelian 2-forms expected to arise on stacks of M5-branes, where they should couple to the endstrings of open membranes.

A closer examination of the above loop-space connections reveals certain features that are known from the theory of **2-groups**, which are a categorified (stringified) version of an ordinary group. This indicates that these constructions have to be thought of as arising in a theory of **categorified gauge theory**. And indeed, this turns out to be the case. The deeper investigation of the above structure requires however to step back and look at the larger picture that is emerging here. This is the content of §3.

3. Nonabelian Strings

We begin our overview of nonabelian strings by give a pedagogical introduction to the concept of 2-group (which is well known to algebraists but hardly known among physicists) in §3.1 (p.33). Then we give an overview of the new results concerning the 2-group which is related to spinning strings, summarizing §10 (p.200). In §3.2 (p.46) the definition of a principal 2-bundle, following [36], and the derivation of the basic cocycle condition is discussed. The main definitions and results of the theory of 2-bundles with 2-holonomy are then given in §3.3 (p.52), summarizing the discussion in §11 (p.242) and §12 (p.285). Finally an overview of the differential approach to these issues is given in §3.4 (p.60), summarizing §13 (p.335).

3.1 2-Groups, Loop Groups and the String-Group

The concept of a 2-group is a basic ingredient for all of the dicussion to follow. It is in principle well-known and well-understood, and, at least for the case of strict 2-groups, which we will mostly make use of, easy to deal with. Before discussing results about "nonabelian strings", i.e. about nonabelian surface holonomy, it should be worthwhile to give the non-expert reader an accessible introduction to the essence of the concept. This is the aim of the following subsection.

3.1.1 Heuristic Motivation of 2-Groups

For illustration purposes, first consider the case of ordinary lattice gauge theory, where one is looking at a graph whose edges are labeled by group elements of some possibly non-abelian group G. These group elements specify a holonomy of some G-connection along the given edge. In order to compute the holonomy associated with a concatenation of elementary edges one simply multiplies the associated group elements in the given order. Due to the associativity of the group product, the total holonomies obtained this way are well-defined in that they do not depend on which edges were concatenated first and which later.

This may seem quite trivial, as certainly it is, but it contains in it the seed of a non-trivial generalization to higher order holonomies.

Suppose we have not just a graph but a 2-complex and not just edges are labeled with group elements, but faces, are, too. (Assume for the moment, for simplicity of exposition, that both, edges and faces, are labeled by elements of the same group G.) The group label of any elementary face can naturally be addressed as the *surface holonomy* of that face.

Is there a way, in analogy to the above line holonomies, that we can associate a total surface holonomy to a connected collection of elementary faces?

It is immediately clear that the associativity of the group product, which is inherently linear in nature, alone is no longer sufficient to capture the "2-associativity" implicit in the different ways in which elementary faces can be composed. Given a square of four faces, for instance, we can first glue them horizontally along their vertical boundaries and then vertically, along their horizontal boundaries – or the other way around. The resulting total surface is of course the same in both cases, but when the group G is not abelian there

is obviously no equally unique way to associate with it a product of the respective four surface labels.

On the other hand, if we just had, say, vertical composition of faces in a linear fashion, there would be no problem. In that case we could just multiply the associated group elements in the respective order.

$$
\begin{tikzcd}
\bullet \arrow[r, bend left=60, "g_1"{name=U}] \arrow[r, "g_2"{name=M}] \arrow[r, bend right=60, "g_3"'{name=D}] & \bullet
\arrow[Rightarrow, from=U, to=M, "f_1"] \arrow[Rightarrow, from=M, to=D, "f_2"]
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\bullet \arrow[r, bend left=60, "g_1"{name=U}] \arrow[r, bend right=60, "g_3"'{name=D}] & \bullet
\arrow[Rightarrow, from=U, to=D, "(f_1 \circ f_2)"]
\end{tikzcd}
$$

We write this *vertical product* of surface elements as

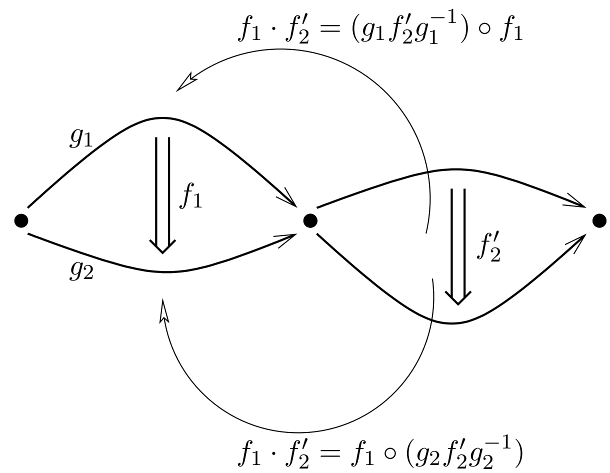
$$
\begin{pmatrix} f_1 \\ \circ \\ f_2 \end{pmatrix} \equiv f_1 \circ f_2 \;\equiv\; f_2 f_1 \,. \tag{3.1}
$$

(Note that here and elsewhere we follow the convention popular in category theoretic literature of writing the composition of arrows $f_1 \circ f_2$ in *literal order* instead of the other way around.) On the right we here have the ordinary product in the group G. The order of the factors is purely conventional and could have been choosen the other way around.

With a vertical product in hand, the task of finding a consistent definition for general surface composition can be solved by defining a consistent way by which horizontally added faces are *inserted* into the vertical string of faces. In other words, a procedure is needed which allows to consistently "squash" a collection of elementary faces until it becomes a linear "vertical" string of faces whose surface holonomies can be multiplied unambiguously.

The "squashing" involves moving surface group labels along the edges of the 1-complex, and this is naturally described by "parallel transporting" them with respect to the edge holonomies. So if we move a surface label f along a directed edge labeled by g, it should become $g^{-1} f g$.

This means that given two horizontally adjacent surface elements with group labels f_1 and f_2' and an edge g_1 along the upper boundary of f_1 to f_2', as well as an edge g_2 along the lower boundary of f_1 to f_2',



we can form the *horizontal* product $f_1 \cdot f_2'$ of f_1 and f_2' by

- *either* first moving f_2' along g_1^{-1} *upper* boundary of f_1 (such that the target edge of f_2' coincides with the source edge of f_1) where it becomes $g_1 f_2' g_1^{-1}$ and where it can be vertically multiplied with f_1 to produce $(g_1 f_2' g_1^{-1}) \circ f_1 = f_1\ g_1 f_2' g_1^{-1}$.
- *or* first moving f_2' along g_2^{-1} to the *lower* boundary of f_1, where it becomes $g_2 f_2' g_2^{-1}$ and where it is vertically multiplied with f_1 in the order $f_1 \circ (g_2 f_2' g_2^{-1}) = g_2 f_2' g_2^{-1}\ f_1$.

In order that the total resulting surface holonomy be well defined, both these results have to agree, which gives a crucial *consistency condition* on the group labels of edges and surfaces:

$$f_1\ g_1 f_2' g_1^{-1} = g_2 f_2' g_2^{-1}\ f_1\,. \tag{3.2}$$

This is fulfilled when the source edge g_1 and the target edge g_2 of f_1 are related by

$$g_2 = f_1 g_1\,.$$

(There can be more general solutions. But only this one leads to the full structure of a 2-group, as explained in the next section.) When this condition is satisfied the computation of total surface holonomy of a collection of elementary faces is independent of the order in which vertical (3.1) composition and horizontal composition

$$f_1 \cdot f_2' \equiv f_1\ g_1 f_2' g_1^{-1} \tag{3.3}$$

is applied, and hence in this case we can associate a well-defined surface holonomy to a collection of elementary faces.

It is helpful to think of this conditions as expressing a higher order form of ordinary associativity (which ensures well defined line holonomies), that we could call *2-associativity*.

Note that both horizontal and vertical products are associative by themselves. For the vertical product this is just the associativity of the group product, while for the horizontal product it is not quite as trivial but can be easily checked. But in both cases this is a *linear* (1-dimensional) notion of associativity.

In order to see how (3.2) encodes a 2-dimensional notion of associativity, consider computing the total surface holonomy of four faces f_1, f'_1, f_2 and f'_2, composed vertically *and* horizontally

$$\begin{tikzcd}[column sep=huge]
\bullet \arrow[r, bend left=60, "g_1"] \arrow[r, "g_2"] \arrow[r, bend right=60, "g_3"'] \arrow[r, phantom, shift left=3ex, "\Downarrow f_1"] \arrow[r, phantom, shift right=3ex, "\Downarrow f_2"] & \bullet \arrow[r, bend left=60, "g'_1"] \arrow[r, "g'_2"] \arrow[r, bend right=60, "g'_3"'] \arrow[r, phantom, shift left=3ex, "\Downarrow f'_1"] \arrow[r, phantom, shift right=3ex, "\Downarrow f'_2"] & \bullet
\end{tikzcd}$$

The fact that the order of composing these faces is irrelevant is expressed by the equation

$$(f_1 \circ f_2) \cdot \left(f'_1 \circ f'_2\right) = \left(f_1 \cdot f'_1\right) \circ \left(f_2 \cdot f'_2\right) , \tag{3.4}$$

which is the form in which the 2-associativity condition usually appears in the 2-group literature (where it is called the 'exchange law'). It is instructive to emphasize the 2-dimensional character of this equation by actually writing the vertical product along the vertical as in (3.1), so that (3.4) becomes

$$\begin{pmatrix} f_1 \\ \circ \\ f_2 \end{pmatrix} \cdot \begin{pmatrix} f'_1 \\ \circ \\ f'_2 \end{pmatrix} = \begin{matrix} (f_1 \cdot f'_1) \\ \circ \\ (f_2 \cdot f'_2) \end{matrix} \quad . \tag{3.5}$$

It is easily checked by using (3.1) and (3.3) that this is equivalent to the relation (3.2) which we used before.

More generally, edges and surfaces need not be labeled by elements of the same group G. We can assume that, while edges are labeled with elements of G, surfaces are labeled with elements of a group H. In order to generalize the definition of the horizontal product to this case we need an action of G on H which mimics the adjoint action of G on itself. Furthermore, in order to generalize the relation between the source and the target edge, one needs a way to send an element of H to an element of G.

The structure needed is known as a **crossed module** (G, H, α, t) of two groups G and H. Here

$$\alpha \colon G \to \mathrm{Aut}(H)$$

is a group homomorphism from G to the automorphisms of H and

$$t \colon H \to G$$

is a homomorphism from H to G. The horizontal product in this more general case then reads

$$f_1 \cdot f_2 = f_1 \alpha(g_1)(f_2)$$

and the relation between the source and the target edge becomes

$$g_2 = t(f_1)\, g_1 \,.$$

In order for all this to be consistent there are the following two compatibility conditions between α and t:

$$\begin{aligned} \alpha(t(h))(h') &= hh'h^{-1} \\ t(\alpha(g)(h)) &= gt(h)\, g^{-1} \,, \end{aligned}$$

which express the idea that $\alpha(g)$ is a generalization of conjugation by g.

3.1.2 2-Groups as Categorified Groups

The above discussion, emphasizing the idea that the horizontal product involves parallel transport of surface labels along edges, gives a rough heuristic approach to 2-groups and their role in 2-holonomy theory. But more formally 2-groups arise as the **categorification** of the concept of an ordinary group. Since the inner workings of 2-groups are important for much of the discussion to follow, and since their derivation nicely illustrates the concept of categorification, we here want to spell this out in detail.

The following makes use of categories and functors between categories. The reader unfamiliar with these concepts is urged to skip to §4.3 (p.72) where a brief introduction to basic elements of category theory is provided.

An ordinary group is defined to be a *set* G together with *functions*

$$G \xrightarrow{s} G$$

(inversion) and

$$G \times G \xrightarrow{m} G \,,$$

(multiplication) which satisfy the *equations*

$$m(g, s(g)) = 1 = m(s(g)\,, g)$$

and

$$m(s_1, m(s_2, s_3)) = m(m(s_1, s_2)\,, s_3) \,.$$

Using the dictionary discussed in §1.1.4.1 (p.14) this is categorified by saying that there is a *category* $\mathcal{G}$ together with a *functor*

$$\mathcal{G} \times \mathcal{G} \xrightarrow{m} \mathcal{G}$$

such that the above equations become natural isomorphisms.

The special case where all these natural isomorphisms are actually identities is called the *strict* case. A *strict 2-group* is hence a category with a product functor as above which satisfies the usual axioms of a group "on the nose".

By going through the above axioms of a strict 2-group $\mathcal{G}$ one can work out how it is described *in terms of two ordinary groups*:

First of all consider all the identity morphisms going from an object $g \in \mathcal{G}$ to itself: $g \xrightarrow{\mathrm{Id}} g$. Restricted to these the axioms for the product functor $m : \mathcal{G} \to \mathcal{G}$ reduce to the axioms of an ordinary group product. Call this group G. Hence for every element in G there is an object in $\mathcal{G}$ and the product between the corresponding identity morphisms is

$$\begin{array}{c} g \\ 1\big\downarrow \\ g \end{array} \cdot \begin{array}{c} g' \\ 1\big\downarrow \\ g' \end{array} = \begin{array}{c} gg' \\ 1\big\downarrow \\ gg' \end{array} ,$$

where we indicate the product functor m by a dot '$\cdot$'.

Next consider the nontrivial morphisms which start at the identity element $1 \in G$, i.e. which are of the form $1 \xrightarrow{f} g$. Obviously, these form a group under the product m themselves, since the product of any two of them is a again a morphism starting at the identity. Call this group H and write

$$\begin{array}{c} 1 \\ h\big\downarrow \\ g \end{array} \cdot \begin{array}{c} 1 \\ h'\big\downarrow \\ g' \end{array} = \begin{array}{c} 1 \\ hh'\big\downarrow \\ gg' \end{array} ,$$

where $h, h' \in H$

Given any morphism $g \xrightarrow{f} g'$, let t denote the operation of sending it to its *target* object, i.e.

$$t\Big(g \xrightarrow{f} g'\Big) \equiv g' .$$

Applying this to the above equation shows that t restricts on those morphisms that start at the identity object to a group homomorphism

$$t \colon H \to G .$$

We can conjugate every morphism in H with an arbitrary identity morphisms and stay in H:

$$\begin{array}{c} g \\ 1\big\downarrow \\ g \end{array} \cdot \begin{array}{c} 1 \\ h\big\downarrow \\ t(h) \end{array} \cdot \begin{array}{c} g^{-1} \\ 1\big\downarrow \\ g^{-1} \end{array} \equiv \begin{array}{c} 1 \\ \alpha(g)(h)\big\downarrow \\ gt(h)\,g^{-1} \end{array} .$$

Since this is just conjugation in our 2-group it obviously gives an automorphism of H and hence the α appearing in the above formula is a group homomorphism from G to $\mathrm{Aut}(H)$:

$$\alpha \colon G \to \mathrm{Aut}(H) .$$

The homomorphisms t and α have to satisfy certain compatibility conditions. The first of these is

$$t(\alpha(g)(h)) = gt(h)\,g^{-1} ,$$

which follows immediately from the above considerations. The other one is

$$\alpha(t(h))\big(h'\big) = hh'h^{-1}\,.$$

This is a consequence of the fact that the multiplication m in the 2-group is a functor. For consider the left-hand side, which is given by

$$\begin{array}{c} 1 \\ \alpha(t(h))(h')\Big\downarrow \\ t\big(hh'h^{-1}\big) \end{array} \qquad = \begin{array}{c} t(h) \\ 1\Big\downarrow \\ t(h) \end{array} \cdot \begin{array}{c} 1 \\ h'\Big\downarrow \\ t(h') \end{array} \cdot \begin{array}{c} t(h)^{-1} \\ 1\Big\downarrow \\ t(h)^{-1} \end{array} .$$

Since the product functor has to respect the composition of morphisms, we can extend the diagram on the right by an identity morphism as follows:

$$\begin{array}{c} t(h) \\ 1\Big\downarrow \\ t(h) \end{array} \cdot \begin{array}{c} 1 \\ h'\Big\downarrow \\ t(h') \end{array} \cdot \begin{array}{c} t(h)^{-1} \\ 1\Big\downarrow \\ t(h)^{-1} \end{array} = \begin{array}{ccccc} 1 & & 1 & & 1 \\ h\Big\downarrow & \cdot & 1\Big\downarrow & \cdot & h^{-1}\Big\downarrow \\ t(h) & & 1 & & t(h)^{-1} \\ 1\Big\downarrow & \cdot & h'\Big\downarrow & \cdot & 1\Big\downarrow \\ t(h) & & t(h') & & t(h)^{-1} \end{array} .$$

Composing these morphisms before multiplying them then yields

$$\cdots = \begin{array}{c} 1 \\ h\Big\downarrow \\ t(h) \end{array} \cdot \begin{array}{c} 1 \\ h'\Big\downarrow \\ t(h') \end{array} \cdot \begin{array}{c} 1 \\ h^{-1}\Big\downarrow \\ t(h)^{-1} \end{array} = \begin{array}{c} 1 \\ hh'h^{-1}\Big\downarrow \\ t\big(hh'h^{-1}\big) \end{array} .$$

This is equivalent to the above consistency condition.

Now we can generalize to arbitrary morphisms. Due to the group structure on our category $\mathcal{G}$, every morphism can be written as a morphism $1 \xrightarrow{h} t(h)$ starting at the identity element and multiplied (from the right, say) with an identity morphism on an object g. We will denote these morphisms by pairs (g, h):

$$(g,h) \equiv \begin{array}{c} g \\ h\Big\downarrow \\ t(h)\,g \end{array} \equiv \begin{array}{c} 1 \\ h\Big\downarrow \\ t(h) \end{array} \cdot \begin{array}{c} g \\ 1\Big\downarrow \\ g \end{array} .$$

Given this definition and what we already know about conjugation in our 2-group, it is

easy to work out the product of general morphisms as follows:

$$
\begin{aligned}
\begin{matrix} g \\ h\big\downarrow \\ t(h)\,g \end{matrix} \cdot \begin{matrix} g' \\ h'\big\downarrow \\ t(h')\,g' \end{matrix}
&= \begin{matrix} 1 \\ h\big\downarrow \\ t(h) \end{matrix} \cdot \begin{matrix} g \\ 1\big\downarrow \\ g \end{matrix} \cdot \begin{matrix} 1 \\ h'\big\downarrow \\ t(h') \end{matrix} \cdot \begin{matrix} g' \\ 1\big\downarrow \\ g' \end{matrix} \\
&= \begin{matrix} 1 \\ h\big\downarrow \\ t(h) \end{matrix} \cdot \begin{matrix} g \\ 1\big\downarrow \\ g \end{matrix} \cdot \begin{matrix} 1 \\ h'\big\downarrow \\ t(h') \end{matrix} \cdot \begin{matrix} g^{-1} \\ 1\big\downarrow \\ g^{-1} \end{matrix} \cdot \begin{matrix} g \\ 1\big\downarrow \\ g \end{matrix} \cdot \begin{matrix} g' \\ 1\big\downarrow \\ g' \end{matrix} \\
&= \begin{matrix} 1 \\ h\big\downarrow \\ t(h) \end{matrix} \cdot \begin{matrix} 1 \\ \alpha(g)(h')\big\downarrow \\ gt(h')\,g^{-1} \end{matrix} \cdot \begin{matrix} gg' \\ 1\big\downarrow \\ gg' \end{matrix} \\
&= \begin{matrix} gg' \\ h\alpha(g)(h')\big\downarrow \\ t(h)\,gt(h')\,g' \end{matrix} .
\end{aligned}
$$

Hence we find the rule for horizontal multiplication

$$(g,h) \cdot (g',h') = (gg', h\alpha(g)\big(h'\big)) \,.$$

This is the multiplication operation in the *semidirect product* of groups $G \ltimes H$, which we have interpreted in terms of parallel transport in the previous subsection §3.1.1 (p.33).

Finally we need to work out what the result of *composing* two morphisms is. For this we again need to make use of the fact that the product is a functor and that it respects the composition of morphisms.

Starting with the composition

$$
\begin{matrix} g \\ h\big\downarrow \\ t(h)\,g \\ h'\big\downarrow \\ t(h'h)\,g \end{matrix}
$$

we can horizontally split this to obtain

$$
\cdots = \begin{matrix} 1 \\ 1\big\downarrow \\ 1 \\ h'\big\downarrow \\ t(h') \end{matrix} \cdot \begin{matrix} g \\ h\big\downarrow \\ t(h)\,g \\ 1\big\downarrow \\ t(h)\,g \end{matrix}
$$

and then use vertical composition to get

$$
\cdots = \begin{matrix} 1 \\ h'\big\downarrow \\ t(h') \end{matrix} \cdot \begin{matrix} g \\ h\big\downarrow \\ t(h)\,g \end{matrix} .
$$

Performing the product operation now yields

$$\cdots = \begin{array}{c} g \\ h'h \Big\downarrow \\ t(h'h)\, g \end{array} .$$

Hence the vertical composition of the morphism (g, h) with the morphism $(t(h)\, g, h')$ is simply the morphism $(g, h'h)$.

Given the concept of a *2-category*, which is briefly discussed in §4.3.2 (p.78), it is clear that we can think of a 2-group as a 2-category with a single object $\bullet$. This is essentially just a consequence of how we can think of an ordinary group as a category with a single object, as explained in §4.3.1.1 (p.73).

So, where we had an object g in the above disucssion, we can think of it as a morphism

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g"] & \bullet \end{tikzcd}$$

starting and ending at the single object $\bullet$. When doing so the morphisms

$$\cdots = \begin{array}{c} g \\ h \Big\downarrow \\ g' \end{array}$$

become 2-morphisms

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g"{name=U}] \arrow[r, bend right, "g'"'{name=D}] & \bullet \arrow[from=U, to=D, Rightarrow, "h"'] \end{tikzcd}$$

and the product functor

$$\begin{array}{c} g_1 \\ h_1 \Big\downarrow \\ g_1' \end{array} \cdot \begin{array}{c} g_2 \\ h_2 \Big\downarrow \\ g_2' \end{array} .$$

becomes nothing but the *horizontal composition* of such 2-morphisms

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g_1"{name=U1}] \arrow[r, bend right, "g_1'"'{name=D1}] & \bullet \arrow[r, bend left, "g_2"{name=U2}] \arrow[r, bend right, "g_2'"'{name=D2}] & \bullet \arrow[from=U1, to=D1, Rightarrow, "h"'] \arrow[from=U2, to=D2, Rightarrow, "h_2"'] \end{tikzcd} .$$

The functoriality of the product, i.e. the fact that it respects (vertical) composition is then nothing but the *exchange law* of 2-categories, which says that the order of horizontal and vertical compositon in diagrams like

$$\begin{tikzcd} a \arrow[r, bend left=60, "g_1"{name=U1}] \arrow[r, "g_2"{name=M1}] \arrow[r, bend right=60, "g_3"'{name=D1}] & b \arrow[r, bend left=60, "g_1'"{name=U2}] \arrow[r, "g_2'"{name=M2}] \arrow[r, bend right=60, "g_3'"'{name=D2}] & c \arrow[from=U1, to=M1, Rightarrow, "f_1"] \arrow[from=M1, to=D1, Rightarrow, "f_2"] \arrow[from=U2, to=M2, Rightarrow, "f_1'"] \arrow[from=M2, to=D2, Rightarrow, "f_2'"] \end{tikzcd}$$

is irrelevant.

It is this 2-categorical language that we will mostly use when talking about 2-groups.

3.1.3 Lie 2-Algebras

Just like an ordinary Lie group has a Lie algebra, a Lie 2-group has a **Lie 2-algebra**, the categorification of an ordinary Lie algebra. We need some basic understanding of Lie 2-algebras for the discussion of the 2-group realization of the String-group below in §3.1.4 (p.43) as well as for the differential formalism to be introduced in §3.4 (p.60).

The theory of Lie 2-algebras has been worked out in [40]. By performing the process of categorification by internalization (§9.1 (p.182)), which is the precise formulation of the categorification dictionary in §1.1.4.1 (p.14), one finds that a (semistrict) Lie 2-algebra $\mathcal{L}$ is a category whose objects x are elements of a vector space V_0 and whose morphisms $x \xrightarrow{\vec{f}} y$ are labeled by elements $\vec{f}$ of another vector space V_1.

In order to emphasize the relation between Lie 2-algebras and Lie 2-groups we could draw such a morphism as follows:

$$\begin{tikzcd} \bullet \arrow[r, bend left, "x"{name=U}] \arrow[r, bend right, "y"'{name=D}] & \bullet \arrow[from=U, to=D, Rightarrow, "\vec{f}"] \end{tikzcd}$$

The source x and target y of this morphism are related by a map

$$d \colon V_1 \to V_0$$

as follows:

$$y = x + d\vec{f}\,.$$

Therefore we can specify any Lie 2-algebra morphism by a couple $(x, \vec{f})$, where x is its source and $\vec{f}$ is called its **arrow part**.

Composition of such morphisms turns out to be given simply by the addition of their arrow parts:

$$(x, \vec{f}) \circ (x + d\vec{f}, \vec{g}) = (x, \vec{f} + \vec{g})\,.$$

So far this data defines a **2-vector space**. A Lie 2-algebra is a 2-vector space with extra structure, the **Lie bracket functor**.

$$[\cdot,\cdot] \colon \mathcal{L} \times \mathcal{L} \to \mathcal{L}\,.$$

This turns out to be expressible in terms of a linear map

$$\begin{aligned} l_2 &: V_0 \times V_0 \to V_0 \\ l_2 &: V_0 \times V_1 \to V_1 \end{aligned}$$

which is antisymmetric on $V_0 \times V_0$ and is given by

$$\Big[(x_1, \vec{f}_1), (x_2, \vec{f}_2)\Big] = \Big(l_2(x_1, x_2)\,,\ l_2\Big(x_1, \vec{f}_2\Big) - l_2\Big(x_2, \vec{f}_1\Big) + l_2\Big(d\vec{f}_1, \vec{f}_2\Big)\Big)\ .$$

Note how this is like a bracket operation on V_0 together with a bracket operation on V_1 which is 'twisted' by elements of V_0.

In the special case that l_2 is a Lie bracket on V_0, that $l_2(d\cdot, \cdot)$ is a Lie bracket on V_1 and that V_0 acts on V_1 by derivations, all this data defines a **differential crossed module**, and $\mathcal{L}$ is the Lie 2-algebra of a *strict* Lie 2-group and is itself called a strict Lie 2-algebra.

More general Lie 2-algebras of the above form are weaker than that and are called *semistrict.* For them the map l_2 fails to satisfy the Jacobi identity, which would read

$$l_2(x, l_2(y, z)) + l_2(z, l_2(x, y)) + l_2(y, l_2(z, x)) = 0\,.$$

This failure is measured by a trilinear antisymmetric map

$$l_3 \colon V_0^3 \to V_1$$

as follows:

$$l_2(x, l_2(y, z)) + l_2(z, l_2(x, y)) + l_2(y, l_2(z, x)) = dl_3(x, y, z)\ .$$

In this sense the strictness property of a Lie algebra is weakened. But this weakening requires a consistency condition, a coherence law. This somewhat intricate law is discussed below in §10.2.1 (p.205) and in more detail in [40] and here we will not bother to write it down.

Given the above definitions it can be easily shown that every strict Lie 2-group has its strict Lie 2-algebra.

Just like a strict 2-group is defined by a crossed module (G, H, α, t) of groups (*cf.* end of §3.1.1 (p.33)), a strict Lie 2-algebra is defined by a differential crossed module $(\mathfrak{g}, \mathfrak{h}, d\alpha, dt)$.

But it turns out to be quite nontrivial to find explicit realizations of semistrict Lie 2-algebras that are not strict and, moreover, to find which, if any, weak 2-group these come from.

3.1.4 The 2-Group $\mathcal{P}_k G$ and its Relation to $\mathrm{String}(n)$

Baez and Crans have introduced in [40] a familiy of non-strict Lie 2-algebras $\mathfrak{g}_k$ for every ordinary Lie algebra $\mathfrak{g}$ which are only very slightly non-strict. Interestingly, even though the weakening in this case superficially looks trivial, it turns out that these Lie 2-algebras $\mathfrak{g}_k$ give rise to the highly non-trivial 2-groups $\mathcal{P}_k G$.

They are defined as follows:

Let k be any real number. The vector space of objects of the Lie 2-algebra $\mathfrak{g}_k$ is simply that of the Lie algebra $\mathfrak{g}$ itself

$$V_0 = \mathfrak{g}\,.$$

The vector space of morphism is just the 1-dimensional one over the real numbers

$$V_1 = \mathbb{R}\,.$$

The map $V_1 \xrightarrow{d} V_0$ is defined to be trivial

$$d(\mathbb{R}) = 0$$

as is the action of V_0 on V_1:

$$l_2(V_0, V_1) = 0\,.$$

The interesting aspect of this is that, due to the triviality of d, the above conditions say that l_2 must be the ordinary Lie bracket on $V_0 = \mathfrak{g}$

$$l_2(x,y) = [x,y]\ , \qquad \forall\, x,y \in \mathfrak{g}\,,$$

even though l_3 may be nontrivial. By solving the coherence law for l_3 in the present case one finds that one can indeed choose it to be non-trivial by setting

$$l_3(x,y,z) = k\langle x, [y,z]\rangle\,.$$

Here $\langle\cdot,\cdot\rangle$ is the Killing form on $\mathfrak{g}$ with respect to some normalization. The real number k appearing here is the one parameterizing the family $\mathfrak{g}_k$ of Lie 2-algebras.

Despite its simplicity, it turns out that $\mathfrak{g}_k$ does not exponentiate to a Lie 2-group [41].

But since $\mathfrak{g}_k$ is a category, not just a set, one has to take care of the following issue: Any two sets with additional structure are 'essentially equal' if they are isomorphic in a way that respects this structure. This is because sets live in the 1-category Set. But categories themselves live in the 2-category Cat. For them to be 'essentially equal' they don't have to be isomorphic but have to be just what is called **equivalent as categories**. This is explained in more detail in §4.3.2 (p.78).

A mere isomorphism between two categories would amount to having two functors between these categories which are mutually inverse. An equivalence of these two categories means that these two functors are not necessarily inverse, but that their composition is naturally isomorphic to the identity functor. This is one level weaker than the concept of direct isomorphism.

For this reason it makes good sense to search for Lie 2-algebras which are *equivalent* in the category-theoretic sense to $\mathfrak{g}_k$.

By some general reasoning one can find that a promising candidate for such an equivalent Lie 2-algebra is the *strict* Lie 2-algebra which comes from the crossed module

$$\mathcal{P}_k\mathfrak{g} = (P_0\mathfrak{g}, \widehat{\Omega_k\mathfrak{g}}, d\alpha, dt)\,.$$

Here $P_0\mathfrak{g}$ is the Lie algebra of smooth based paths in $\mathfrak{g}$ with pointwise Lie bracket, while $\widehat{\Omega_k\mathfrak{g}}$ is the Kac-Moody centrally extended Lie algebra of smooth based loops in $\mathfrak{g}$ with pointwise Lie bracket and $d\alpha$ and dt are the natural operations on these.

One can then indeed check that the infinite-dimensional Lie 2-algebra $\mathcal{P}_k\mathfrak{g}$ is equivalent to $\mathfrak{g}_k$. This result involves some rather nontrivial numerical 'coincidences' and is presented in detail in §10.5 (p.226). Interestingly, the nontrivial l_3 that is present in $\mathfrak{g}_k$ is related to boundary terms of the Kac-Moody cocycle coming from the paths in $P_0\mathfrak{g}$.

In other words, this result says that we can understand the non-strict Lie 2-algebra $\mathfrak{g}_k$ as being that Lie 2-algebra which is obtained by starting with the strict but 'large' Lie 2-algebra $\mathcal{P}_k\mathfrak{g}$ and then taking isomorphism classes of objects.

The relevance of this result is that the Lie 2-algebra $\mathcal{P}_k\mathfrak{g}$, being strict, does have a corresponding strict Lie 2-group – but only if k is integer. This latter condition is a

consequence of the infinite-dimensionality of $\mathcal{P}_k\mathfrak{g}$. But it is in fact nothing but the well-known condition that the *level* k of a Kac-Moody centrally extended loop group has to be an integer.

The strict Lie 2-group corresponding to $\mathcal{P}_k\mathfrak{g}$ is called $\mathcal{P}_kG$ and is given by the crossed module

$$\mathcal{P}_kG = (P_0G, \widehat{\Omega_k G}, \alpha, t)\,,$$

where P_0G is the group of smooth based paths in G and $\widehat{\Omega_k G}$ is the Kac-Moody centrally extended group of smooth based loops in G. The details of how this group can be constructed are reproduced in §10.4.1 (p.219).

Following the above given motivation, it is now interesting to see how, for the case $G = \mathrm{Spin}(n)$, the 2-group $\mathcal{P}_kG$ 'is' (related to) the group $\mathrm{String}(n)$.

This requires some explanation, but in a simple form it can already be made plausible as follows:

First note that there is a simple way in which we can construct a Lie 2-algebra $\mathrm{b}\mathfrak{u}(1)$ from $\mathfrak{u}(1)$ by 'lifting' $\mathfrak{u}(1)$ to the space of morphisms, i.e. by letting $V_0 = 0$ and $V_1 = \mathfrak{u}(1) \simeq i\mathbb{R}$. Similarly, we can regard any ordinary Lie algebra $\mathfrak{g}$ as a Lie 2-algebra by setting $V_0 = \mathfrak{g}$ and $V_1 = 0$.

There is a morphism of Lie 2-algebras $\mathfrak{g}_k \to \mathfrak{g}$ which just forgets about the label of the morphisms in $\mathfrak{g}_k$. Similarly, there is a morphism of Lie 2-algebras $\mathrm{b}\mathfrak{u}(1) \to \mathfrak{g}_k$ which is just the injection of morphisms based at $x = 0$. Obviously we have an exact sequence

$$0 \to \mathrm{b}\mathfrak{u}(1) \to \mathfrak{g}_k \to \mathfrak{g} \to 0\,.$$

This simple sequence can be thought of, in a sense to be made precise shortly, as the infinitesimal version of the sequence of groups mentioned before:

$$1 \to K(\mathbb{Z},2) \to \mathrm{String}(n) \to \mathrm{Spin}(n) \to 1$$

(for the case that $\mathfrak{g} = \mathrm{Lie}(\mathrm{Spin}(n))$). This means that $\mathfrak{g}_k$ is in a certain sense the infinitesimal version of the sought-after group $\mathrm{String}(n)$.

Essentially the same statement can be made as follows: If we think of the ordinary group G as a 2-group with just identity morphisms, then there is an obvious morphism of 2-groups $\mathcal{P}_kG \xrightarrow{\pi} G$ which sends every path in P_0G to its endpoint and just forgets about the morphisms. This map has as strict kernel the 2-group called $\mathcal{L}_kG$ which is the sub-2-group of $\mathcal{P}_kG$ where all objects are closed paths. Hence there is a strictly exact sequence of 2-groups

$$1 \to \mathcal{L}_kG \xrightarrow{\iota} \mathcal{P}_kG \xrightarrow{\pi} G \to 1\,.$$

This sequence again can be identified in a certain sense with the sequence defining the group $\mathrm{String}(n)$ for the case that $\mathfrak{g} = \mathrm{Lie}(\mathrm{Spin}(n))$, which implies that in this sence $\mathcal{P}_kG$ is like $\mathrm{String}(n)$.

The sense in which this is true is the following:

Every topological category C (one whose set of objects and morphisms are topological spaces such that all operations in the category are continuous functions) gives rise to

a simplicial space, called the *nerve* of this category. By using the topology on C this simplicial space can be turned into a topological space called the 'geometric realization' of the simplicial space. This topological space is called $|C|$.

Here is how to think of the space $|C|$: The points in $|C|$ are all the objects of our category C. The edges in $|C|$ are the morphism in C. The triangles in $|C|$ are given by pairs of composable morphsism in C. And so on: $(p+1)$-simplices in $|C|$ are given by p-tuples of composable morphism in C.

When the category C is a topological 2-group, the group multiplication defined on it makes $|C|$ into an ordinary topological group. In fact, the operation $|\cdot|$ is a functor from the category of topological 2-groups to that of topological groups.

Hence when this functor is applied to the above sequence of 2-groups we obtain the sequence

$$1 \to |\mathcal{L}_1 G| \xrightarrow{|\iota|} |\mathcal{P}_1 G| \xrightarrow{\pi} G \to 1$$

of topological groups, where one uses the fact that $|G| = G$.

Finally, from an old result by Segal it follows that $|\mathcal{L}_1 G|$ is in fact the Eilenberg-MacLane space $K(\mathbb{Z}, 2)$ space. As is explained in §4.2 (p.68), this implies that for $G = \mathrm{Spin}(n)$ we have $|\mathcal{P}_1 G| \simeq \mathrm{String}(n)$.

3.2 Principal 2-Bundles

With the concept of 2-group in hand, we can now define and study the categorification of principal bundles so as to obtain 2-bundles with **structure 2-group**.

The process of *categorification* essentially amounts to taking any algebraic structure, expressing it in terms of diagrams which depict morphisms in some 1-category, and then re-interpreting the very same diagrams as depicting morphisms in a suitable 2-category.

The study of categorified bundles, which was initiated by Toby Bartels [36], is, as far as bare bundles without any extra structure and properties are concerned, actually the simplest example of this procedure.

This is because an ordinary bundle is nothing but some space E, called the **total space**, some space B, called the **base space**, together with a map

$$E \xrightarrow{p} B$$

called the projection map. Better yet, E and B are objects in some category C and p is a morphism in that category. This is the simplest "diagram" that one can think of.

For instance, if C is the category of topological spaces, then E and B are any two topological spaces and p is any continuous map from E to B. Or, if C is the category of smooth spaces then E and B are any two smooth spaces and p is a smooth map between them.

If we forget about all extra structure for the moment we can think of C as the category Set whose objects are (small) sets and whose morphisms are nothing but functions between sets. Then E and B are any two sets and p is any function of sets between E and B.

Now, the category Set of all (small) sets has a natural categorification, namely the 2-category Cat of all categories. The objects of Cat are categories, the morphisms of Cat

are functors between categories and the 2-morphisms of Cat are natural transformations between these functors (*cf.* §4.3.2 (p.78)).

This means that if we interpret the diagram

$$E \xrightarrow{p} B$$

as a morphism in Cat it describes

- a category E
- a category B
- a functor $p\colon E \to B$.

This is what is called a **2-bundle**.

Like an ordinary (1-)bundle is just a map of elements of one set E to elements of another set B, a 2-bundle is a map of *morphisms* (together with their source and target objects, of course) of the category E to morphisms of the category B, such that composition of morphisms is respected.

This is easy enough. Now let us add extra properties to the notion of "bundle" and categorify them, too. The bundles that play a role in gauge theory are **locally trivializable fiber bundles** within the category of smooth spaces. This means that they have the property that the pre-image $E|_{U_i} = p^{-1}(U_i)$ of a contractible patch U_i of the smooth base space B is isomorphic, via an isomorphism t_i, to the cartesian product of U_i with another smooth space, F, called the **typical fiber**, such that this diagram commutes:

$$\begin{tikzcd} E|_{U_i} \arrow[rr, "t_i"] \arrow[dr, "p"'] & & U_i \times F \arrow[dl] \\ & U_i & \end{tikzcd} .$$

(Here the arrow on the right is just the projection from $U_i \times F$ onto the U_i factor.)

Let us again think of this as a diagram in Set. The fact that this diagram commutes really means that there is a 2-morphism from the 1-morphism $E|_{U_i} \xrightarrow{t_i} U_i \times F \to U_i$ to the 1-morphism $E|_{U_i} \xrightarrow{p} U_i$. But since Set is a 1-category, all its 2-morphisms are identity 2-morphisms, which implies the intended equality

$$\begin{aligned} & E|_{U_i} \xrightarrow{t_i} U_i \times F \to U_i \\ = \; & E|_{U_i} \xrightarrow{p} U_i \,. \end{aligned}$$

Categorifying this diagram means taking the same diagram, but interpreting it as a diagram in Cat. Then the diagram says that there is a *functor* $E|_{U_i} \xrightarrow{t_i} U_i \times F \to U_i$ and a *functor* $E|_{U_i} \xrightarrow{p} U_i$, since 1-morphisms in Cat are functors between categories.

But Cat, being a 2-category, also has 2-morphisms between its 1-morphisms, given by natural transformations between functors. So in Cat the implcit 2-morphism expressing the commutativity of the above diagram becomes a (invertible) natural transformation λ

$$\begin{aligned} & E|_{U_i} \xrightarrow{t_i} U_i \times F \to U_i \\ \overset{\lambda}{\Rightarrow} \; & E|_{U_i} \xrightarrow{p} U_i \,. \end{aligned}$$

This is how categorification works. Of course, one can now refine this construction by adding additional structure. Usually we want our ordinary bundles to be not just sets with functions between sets, but to be smooth spaces with smooth maps between them. This amounts to interpreting the above diagrams not in Set, but in Diff, the category of smooth spaces, whose objects are smooth spaces and whose morphisms are smooth maps.

If we want to have something similar in the categorified case, we need a 2-category $2C^\infty$, whose objects are "smooth categories", whose morphisms are "smooth functors" and whose 2-morphisms are "smooth natural transformations" between smooth functors.

A smooth category can be defined as a category whose diagrammatic definition is interpreted not in Set, but in Diff. This means that it is a category which has not just a *set* of objects and a *set* of morphisms, but where the objects and morphisms both form a smooth space and where all operations in the category, such as composition of morphisms, is given by smooth maps between smooth spaces. Since such a smooth category can be regarded as the categorification of the concept of a smooth space, it is called a **2-space** [36].

The archetypical example of this is the category obtained from the space of paths in some smooth space U. Objects are all the points of U and morphisms are all the paths of U. Physicists should think of such a category as a configuration space for a (open) string, being the categorification of an ordinary space U, which can be regarded as the configuration space of a particle.

Hence the locally trivializable 2-bundles that we shall be concerned with are given by the diagrams

$$E \xrightarrow{p} B$$

and

$$\begin{aligned} & E|_{U_i} \xrightarrow{t_i} U_i \times F \to U_i \\ \overset{\lambda}{\Rightarrow} \; & E|_{U_i} \xrightarrow{p} U_i \,, \end{aligned}$$

interpreted as diagrams in the 2-category $2C^\infty$ of smooth spaces.

This game can now be played further. Next we will want to consider the categorification of the concept **principal fiber bundle**. A principal fiber bundle is a locally trivializable bundle of the above kind such that the typical fiber F is a Lie group, $F = G$, and such that the transition $g_{ij} \equiv t_i^{-1} \circ t_j|_{U_{ij}}$ from the trivialization over U_i to the trivialization over U_j is given by multiplication in the group G.

We already know from §3.1.2 (p.37) how to categorify this part. Hence a **principal 2-bundle** is a locally trivializable 2-bundle of the above sort such that the typical fiber category is a Lie 2-goup $\mathcal{G}$.

One can now essentially copy all the considerations concerning ordinary principal bundles from the textbook, while contiuously expressing everything in terms of diagrams in Diff and re-interpreting all these diagrams in $2C^\infty$, thus obtaining the theory of principal 2-bundles. Up to some details this amounts essentially to applying the dictionary from §1.1.4.1 (p.14) to all the familiar phenomena.

For instance in ordinary principal bundles we have the transition *functions*

$$g_{ij} \colon U_{ij} \to G$$

from double overlaps of patches of the base space to the structure group G, and these functions satisfy an *equation*

$$g_{ik} = g_{ij} \cdot g_{jk}$$

on triple overlaps U_{ijk}. According to the dictionary in §1.1.4.1 (p.14) a principal 2-bundle will have transition *functors*

$$g_{ij} \colon U_{ij} \to \mathcal{G}$$

from a double intersection of patches of the base *2-space* to the structure 2-group, such that on triple intersections there is a *natural isomorphism* f_{ijk}

$$g_{ik} \stackrel{f_{ijk}}{\Rightarrow} g_{ij} \cdot g_{jk} \,.$$

Functions are replaced by functors, and equations between functions are replaced by natural isomorphisms between functors. The existence of the natural isomorphism f_{ijk} is characterized by the following naturality diagram:

$$\begin{array}{c|c}
U_{ijk} & \mathcal{G} \\
\hline
\begin{tikzcd} x \arrow[d, "\gamma"'] \\ y \end{tikzcd}
&
\begin{tikzcd} g_{ik}(x) \arrow[r, "f_{ijk}(x)"] \arrow[d, "g_{ik}(\gamma)"'] & g_{ij} \cdot g_{jk}(x) \arrow[d, "g_{ij} \cdot g_{jk}(\gamma)"] \\ g_{ik}(y) \arrow[r, "f_{ijk}(y)"'] & g_{ij} \cdot g_{jk}(y) \end{tikzcd}
\end{array}$$

Here $x \xrightarrow{\gamma} y$ is a morphism in the 2-space U_{ijk} which is mapped by g_{ik} to the morphism $g_{ik}(x) \xrightarrow{g_{ik}(\gamma)} g_{ik}(y)$ in the 2-group $\mathcal{G}$. Similarly for $g_{ij} \cdot g_{jk}$, where $\cdot$ denotes the product functor in the 2-group.

We shall mostly be interested in this diagram for the special case that the base 2-space B of our 2-bundle is really just an ordinary space, i.e. a smooth category all of whose morphisms are identity morphisms.

In this case the above naturality diagram reduces to

$$\begin{array}{c|c} U_{ijk} & \mathcal{G} \\ \hline \begin{tikzcd} x \arrow[d, "\mathrm{Id}"'] \\ x \end{tikzcd} & \begin{tikzcd} g_{ik}(x) \arrow[r, "f_{ijk}(x)"] \arrow[d, "\mathrm{Id}"'] & g_{ij}\cdot g_{jk}(x) \arrow[d, "\mathrm{Id}"] \\ g_{ik}(x) \arrow[r, "f_{ijk}(x)"'] & g_{ij}\cdot g_{jk}(x) \end{tikzcd} \end{array} \tag{3.6}$$

Obviously, this diagram commutes in any case. Hence the existence of the natural isomorphism f_{ijk} becomes tantamount to the mere existence of a morphism $g_{ik}(x) \overset{f_{ijk}(x)}{\longrightarrow} g_{ij}\cdot g_{jk}(x)$ in $\mathcal{G}$. If we think of $\mathcal{G}$ as a 2-category with a single object $\bullet$, then the diagram expressing this is the following:

$$\begin{tikzcd} & \bullet \arrow[dr, "g_{jk}"] & \\ \bullet \arrow[ur, "g_{ij}"] \arrow[rr, "g_{ik}"'] \arrow[rr, phantom, shift left=5ex, "\Uparrow f_{ijk}"] & & \bullet \end{tikzcd}$$

From the properties of (strict) 2-groups that were discussed in §3.1.2 (p.37), we know that f_{ijk} is represented by a pair

$$(g_{ik} \in G, f_{ijk} \in H)\,,$$

where G and H are the two groups of the crossed module that describes the 2-group $\mathcal{G}$, and where we conveniently denote the element in H by the same letter as the morphism itself. For this morphism to have $g_{ij}\cdot g_{jk}$ as its target, the relation

$$g_{ij}g_{jk} = t(f_{ijk})\,g_{ik}$$

has to hold. In the context of nonabelian gerbes this is known as one of the cocycle relations describing these structures. It generalizes the ordinary relation $g_{ij}g_{jk} = g_{ik}$ of an ordinary bundle.

But now there must be another equation governing the transition of the $f_{ijk} \in H$ themselves. This is given by a **coherence law** in the 2-bundle.

Recall that categorification implied the weakening of equations to mere natural isomorphisms. We had indicated how this can be understood as replacing identity 2-morphisms in a 1-category like Set with nontrivial 2-morphisms in a 2-category like Cat. But there

are also identity 3-morphisms in Set, which however remain “invisible” since they only go between identity 2-morphisms. But in Cat we can have identity 3-morphisms between non-identity 2-morphisms. These give rise to coherence laws. These laws ensure that the weakening that takes place in categorification is consistent.

In the present case this means the following. Since f_{ijk} is invertible, the above triangular diagram can be read as a prescription for reducing $g_{ij}g_{jk}$ to g_{ik} by using $\bar{f}_{ijk}$, the inverse of f_{ijk}. This is similar to a product operation and we want this operation to be *associative*. In other words, the transformation of $g_{ij}g_{jk}g_{kl}$ to g_{il} on quadruple overlaps U_{ijkl} must be well defined. This is expressed by the following equation:

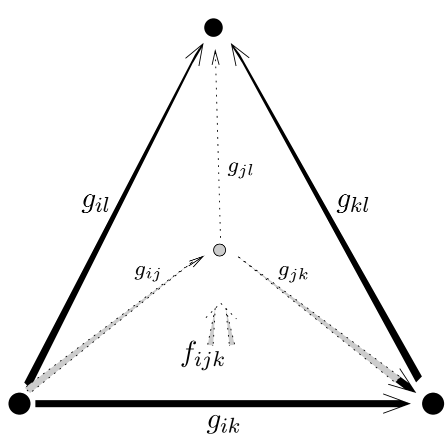
$$\begin{tikzcd}
\bullet \arrow[r, "g_{jk}"] \arrow[dr, phantom, "\Searrow \bar{f}_{ijk}", pos=0.25] \arrow[dr, phantom, "\Downarrow \bar{f}_{ikl}", pos=0.75] & \bullet \arrow[d, "g_{kl}"] & & \bullet \arrow[r, "g_{jk}"] \arrow[dr, "g_{jl}"] & \bullet \arrow[d, "g_{kl}"] \arrow[dl, phantom, "\Swarrow \bar{f}_{jkl}", pos=0.25] \arrow[dl, phantom, "\Downarrow \bar{f}_{ijl}", pos=0.75] \\
\bullet \arrow[u, "g_{ij}"] \arrow[ur, "g_{ik}"] \arrow[r, "g_{il}"'] & \bullet & = & \bullet \arrow[u, "g_{ij}"] \arrow[r, "g_{il}"'] & \bullet
\end{tikzcd}$$

When working out what this equation says in terms of ordinary group elements one finds the relation

$$f_{ijk}f_{jkl} = \alpha(g_{ij})(f_{jkl})\, f_{ijl}\,.$$

This is known in the theory of nonabelian gerbes as the basic cocycle relation for f_{ijk}.

In order to better see what this has to do with identity 3-morphisms, note that the two sides of the above equation can be glued along their common boundary to produce a tetrahedron like this:



The above equation is equivalent to saying that there is a 3-morphism inside this tetrahedron, going between the two 2-morphisms that constitute the boundary of the tetrahedron.

Since we are working with 2-bundles that live in the 2-category $2C^\infty$, this 3-morphism has to be an identity 3-morphism and hence the 2-morphisms on the boundary have to satisfy an equation.

This are the basic ideas concerning principal 2-bundles. Next we discuss how to define 2-connections with 2-holonomy on these 2-bundles. That will allow us to describe nonabelian strings.

3.3 Global 2-Holonomy

In 1985, Alvarez had stated [42], motivated by topological field theory, a procedure for computing global surface holonomy for what would now be called abelian gerbes with connection and curving, or, as we now know, equivalently 2-bundles over ordinary base spaces with structure group given by the crossed module $(G = 1, H = U(1)\,, \alpha = \text{trivial}, t = \text{trivial})$. This formula, which also appeared in [43], uses a covering of base space, works in local patches and glues things appropriately.

This procedure was later found to be precisely the right one to describe the coupling of the fundamental string to the Kalb-Ramond field. In the context of open strings attached to D-branes it is discussed for instance in [44, 45]. In [46] it is shown how this procedure is equivalent to a more intrinsic definition of holonomy of an abelian gerbe as defined for instance in [47, 48].

In order to describe "nonabelian strings" we are interested in a generalization of this formula to more general cases and in an understanding of its mechanism in terms of categorified holonomy in 2-bundles.

There are several equivalent definitions of the notion of an ordinary connection on an ordinary bundle. The categorifications of all these definitions need not be equivalent, however. What we are interested in is a notion of categorified connection (2-connection) that allows to define a notion of categorified parallel transport and categorified holonomy (2-holonomy) and hence a notion of nonabelian strings. Therefore the definition of connection that we want to categorify is that which characterizes a connection as something that allows to do parallel transport. Furthermore, we want to work with local trivializations in order to obtain explicit algorithms for how to compute global surface holonomies from a set of local differential forms on base space.

So the definition of an ordinary connection that is best suited for our needs is the following:

Given a base manifold M with good covering $\mathcal{U} \to M$ and a Lie group G, a **connection on a principal G-bundle** *over M is*

- *on each patch U_i a functor*
$$\mathrm{hol}_i \colon \mathcal{P}_1(U_i) \to G$$
from the groupoid $\mathcal{P}_1(U_i)$ of paths in U_i (cf. §4.3.1.1 (p.73)) to the group G (regarded as a category with a single object),

- *on each double overlap U_{ij} an invertible* morphism of functors

$$\mathrm{hol}_i \xrightarrow{g_{ij}} \mathrm{hol}_j \,,$$

i.e. a natural isomorphism between the restrictions of hol_i *and* hol_j *to U_{ij},*

- *on each triple overlap an equation*

$$g_{ij} \circ g_{jk} = g_{ik}$$

between these natural isomorphisms.

In the following §3.3.1 (p.55) we briefly point out how this does indeed encode the familiar properties of a G-connection.

The above definition has a straightforward elegant categorification. In fact, it is just as easy to categorify it once as to categorify it once again. Therefore we would like to state the general notion of **p-holonomy** in the above sense, for arbitrary integer p.

An n-category has objects, (1-)morphisms between objects, 2-morphism between 1-morphisms, 3-morphisms between 2-morphisms, and so on, up to n-morphisms between $(n-1)$-morphisms (*cf.* §4.3.2.1 (p.81)).

Up to technical details it is obvious what a **p-category of p-paths** in a patch U_i should look like: Its objects should be points in U_i, its morphisms should be paths in U_i, its 2-morphisms should be bounded surfaces in U_i, and so on, up to p-dimensional hypervolumes in U_i. When done correctly, all the n-morphisms in such a category are invertible up to equivalence, and hence we really have a **p-groupoid of p-paths** in U_i. This we shall call $\mathcal{P}_p(U_i)$.

Similarly, like we have 1-groups and 2-groups, we can consider p-groups for general integer p. Let us pick some such p-group and call it G_p, the **structure p-group**. (In fact, the definition of p-holonomy that we are about to give works just as well if G_p is just a p-groupoid.)

A p-connection has to be something that, locally, labels n-dimensional hypervolumes in a patch U_i by some n-morphisms of the structure p-group in a way that is compatible with the composition of such hypervolumes. This is nothing but a p-functor

$$\mathrm{hol}_i \colon \mathcal{P}_p(U_i) \to G_p \,.$$

Two (1-)functors can be related by a morphism of 1-functors (= a natural transformation), if their images are "homotopic", i.e. if their images can be translated into each other inside the target category. Similarly, there are morphisms of this sort between p-functors, sometimes called pseudo-natural transformations. But, for $p > 1$, there can in addition be 2-morphisms between these pseudo-natural transformations, and 3-morphisms between these, and so on, up to p-morphisms between $(p-1)$-morphisms of p-functors.

This gives rise to the following conception of p-connection:

Given a base manifold M with good covering $\mathcal{U} \to M$ and a Lie p-group(oid) G_p, a **p-connection with p-holonomy on a locally trivialized principal G_p-p-bundle** *over M is*

- *on each patch U_i a p-functor*

$$\mathrm{hol}_i \colon \mathcal{P}_p(U_i) \to G_p$$

from the p-groupoid $\mathcal{P}_p(U_i)$ of p-paths in U_i to the p-group(oid) G_p,

- *on each double overlap U_{ij} a* morphism of p-functors

$$\mathrm{hol}_i \xrightarrow{g_{ij}} \mathrm{hol}_j \,,$$

- *on each triple overlap a 2-morphisms of p-functors*

$$g_{ik} \xrightarrow{f_{ijk}} g_{ij} \circ g_{jk} \,,$$

- *in general, on each n-fold overlap an n-morphisms of p-functors between the $(n-1)$-morphisms of p-functors on the respective $(n-1)$-fold intersections, where the n-morphism consitutes the interior and the $(n-1)$-morphisms the faces of an n-simplex.*

Another way to summarize this is to say that, roughly, a locally trivialized principal p-bundle with p-connection is a simplicial map from an abstract p-simplex to the p-category of local holonomy p-functors. See figure 7 (p. 21). This makes it obvious that a **gauge transformation** from one local trivialization to another is nothing but a natural transformation of this simplicial map.

This is the general idea, though we shall mostly be concerned with the cases $p = 1$ and $p = 2$ and just a little bit with $p = 3$. We shall show for $p = 2$ how this definition encodes all the cocycle relations of a p-bundle with p-connection and p-holonomy, and how the local p-functors hol_i glue together to give a globally defined p-holonomy.

The most elegant way to see this is by realizing how the above construction is really the local trivialization of a global p-holonomy p-functor

$$\mathrm{hol} \colon \mathcal{P}_p(M) \to G_p - p\mathrm{Tor}$$

from p-paths in the base manifold M to the p-category of G_p-p-torsors. More precisely, for a given principal G_p-bundle $E \to M$ over a categorically trivial base space M (i.e. for M an ordinary manifold), let $\mathrm{Trans}_p(E)$ be the smooth category whose objects are the fibers E_x, $x \in M$, regarded as G_p-p-torsors and whose n-morphisms are the p-torsor n-morphisms between these. Then a smooth G_p-p-bundle $E \to M$ with p-connection and p-holonomy should be a smooth functor

$$\mathrm{hol} \colon \mathcal{P}_p(M) \to \mathrm{Trans}_p(E) \,.$$

We shall show that this is the case for $p = 1$ and $p = 2$ in §12.4 (p.314).

It turns out that the above notion of p-connection has a *differential* reformulation, which is quite useful and much easier to handle in the general case. This is discussed in §3.4 (p.60).

3.3.1 1-Connections with 1-Holonomy in 1-Bundles

To start with, let us check how the above definition works in the familiar case of 1-bundles.

A connection in an ordinary bundle $E \to M$ locally gives rise to a functor from the groupoid of paths in the base space to the structure group, regarded as a category.

Given a patch $U_i \subset M$, we denote by $\mathcal{P}_1(U_i)$ the groupoid of paths in U_i. The objects of this groupoid are points in U_i, while the morphisms are ***thin* homotopy** equivalence classes of smooth paths between these points. Thin homotopy is homotopy induced by *degenerate surfaces*. Hence dividing out by thin homotopy divides out by reparameterizations of paths and removes *zig-zag moves*, i.e. of pieces of path that retrace themselves. If G is the structure group, regarded as a category with a single object and all morphisms invertible, then any smooth functor

$$\mathrm{hol}_i \colon \mathcal{P}_1(U_i) \to G$$

defines a connection on the trivial bundle $E|_{U_i} \to U_i$. Let us call this functor the local **holonomy (1-)functor** on U_i.

In a well known way the specification of any such functor is equivalent to choosing a 1-form

$$A_i \in \Omega^1(U_i, \mathfrak{g} = \mathrm{Lie}(G)) \ .$$

A gauge transformation is nothing but a natural isomorphism

$$\mathrm{hol}_i \xrightarrow{h} \widetilde{\mathrm{hol}}_i$$

between two such functors. Any such gauge transformation is given by a group-valued function

$$h_i \in C^\infty(U_i, G)$$

and its action on the 1-form A_i coming with hol_i is

$$A_i \to h_i A_i h_i^{-1} + h_i \mathbf{d} h_i^{-1} \ .$$

In particular, the holonomy functors on overlapping patches are related by a gauge transformation induced by the transition function of the local trivialization. Hence if

$$\mathrm{hol}_i \colon \mathcal{P}_1(U_i) \to G$$

and

$$\mathrm{hol}_j \colon \mathcal{P}_1(U_j) \to G$$

are holonomy functors on U_i and U_j, respectively, then their restrictions to the double overlap $U_{ij} = U_i \cap U_j$ are naturally isomorphic

$$\mathrm{hol}_i|_{U_{ij}} \xrightarrow{g_{ij}} \mathrm{hol}_j|_{U_{ij}} \ ,$$

where the natural isomorphism is induced by the transition function

$$g_{ij} \colon U_{ij} \to G \, .$$

In terms of diagrams this means that we have a commuting naturality diagram of the following form:

$$\begin{tikzcd}
\bullet \arrow[r, "g_{ij}(x)"] \arrow[d, "\mathrm{hol}_i([\gamma])"'] & \bullet \arrow[d, "\mathrm{hol}_j([\gamma])"] \\
\bullet \arrow[r, "g_{ij}(y)"'] & \bullet
\end{tikzcd}$$

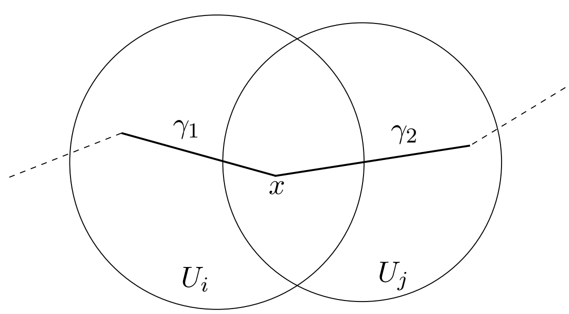
Here $[\gamma]\colon x \to y$ is a morphism in $\mathcal{P}_1(U_{ij})$, namely (the class of) a parameterized path γ in U_{ij}.

By the above, this implies for the connection 1-forms A_i and A_j the relation

$$A_i = g_{ij} A_j g_{ij}^{-1} + g_{ij} \mathbf{d} g_{ij}^{-1} \,.$$

This is called the **transition law** or **cocycle condition** for the connection 1-form.

As is well known, the local holonomy functors hol_i can be glued to give a global holonomy functor. Given any path in the base manifold, this amounts to cutting the path into segments that sit in single patches, computing the local holonomy of the paths in these patches and then gluing them by insertions of the transition function g_{ij}, as indicated in the following figure:



$$\Big\downarrow \,\mathrm{hol}$$

$$\cdots \bullet \xrightarrow{\mathrm{hol}_i(\gamma_1)} \bullet \xrightarrow{g_{ij}(x)} \bullet \xrightarrow{\mathrm{hol}_j(\gamma_2)} \bullet \cdots$$

In formulas this says that the total holonomy is given by the product

$$\cdots \mathrm{hol}_i(\gamma_1)\, g_{ij}(x)\, \mathrm{hol}_j(\gamma_2) \cdots \,.$$

This is gauge invariant because under a gauge transformation we have

$$\begin{aligned} \mathrm{hol}_i\Big(x \xrightarrow{\gamma} y\Big) &\mapsto h_i(x)\,\mathrm{hol}_i(\gamma)\,h^{-1}(y) \\ g_{ij}(x) &\mapsto h_i(x)\,g_{ij}(x)\,h_j^{-1}(x)\ . \end{aligned}$$

There is a deeper principle behind this property, which we will describe in §12 (p.285).

3.3.2 2-Connections with 2-Holonomy in 2-Bundles

We lift this scenario to 2-bundles by categorifying everything in sight.

This requires first of all to define a 2-path 2-groupoid $\mathcal{P}_2 U_i$ on every patch U_i. The objects of $\mathcal{P}_2(U_i)$ are the points $x \in U_i$ and the morphisms are parameterized paths $x \xrightarrow{\gamma} y$ between them. There are now also 2-morphisms

$$[\Sigma] \colon \gamma_1 \to \gamma_2$$

between paths with coinciding endpoints. These 2-morphisms are given by thin homotopy equivalence classes of homotopies between the two given paths, i.e. by equivalence classes of surfaces which interpolate between two given paths.

We already know that, upon categorifying, the structure group G becomes a 2-group $\mathcal{G}$, which we can regard as a 2-category with a single object and with all 1-morphisms and 2-morphisms invertible.

This implies that the categorified local holonomy functors hol_i should be smooth 2-functors from the 2-categories $\mathcal{P}_2(U_i)$ to the structure 2-group:

$$\mathrm{hol}_i \colon \mathcal{P}_2(U_i) \to \mathcal{G}\,.$$

$$\mathrm{hol}_i\left(\begin{tikzcd} x \arrow[r, bend left, "\gamma_1"{name=U}] \arrow[r, bend right, "\gamma_2"'{name=D}] & y \arrow[Rightarrow, from=U, to=D, "{[\Sigma]}"] \end{tikzcd}\right) \equiv \begin{tikzcd} \bullet \arrow[r, bend left, "\mathrm{hol}_i(\gamma_1)"{name=U}] \arrow[r, bend right, "\mathrm{hol}_i(\gamma_2)"'{name=D}] & \bullet \arrow[Rightarrow, from=U, to=D, "\mathrm{hol}_i(\Sigma)"] \end{tikzcd}$$

One expects that these functors are again specified by differential forms on U_i. We will show that, indeed, specifying such hol_i is equivalent to specifying a 1-form

$$A_i \in \Omega^1(U_i, \mathfrak{g})$$

and a 2-form

$$B_i \in \Omega^2(U_i, \mathfrak{h})$$

where $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{h} = \mathrm{Lie}(H)$ are the Lie algebras belonging to the groups G and H that constitute the crossed module (G, H, α, t) coming from the strict 2-group $\mathcal{G}$.

More precisely, we will show that hol_i comes from a family of connection 1-forms

$$\mathcal{A}_i \in \Omega^1\big(P_s^t(U_i)\,, \mathfrak{h}\big)$$

on the spaces $P_s^t(U_i)$ of paths in U_i with endpoints s, t. These are given by the formula

$$\mathcal{A}_i(\gamma) = \int_\gamma \alpha(W_{A_i})(\mathrm{ev}^*(B_i))\ .$$

Here

$$\begin{aligned} \mathrm{ev}\colon P_s^t(U_i) \times [0,1] &\to U_i \\ (\gamma, \sigma) &\mapsto \gamma(\sigma) \end{aligned}$$

is the *evaluation map* which sends a path $\gamma \in P_s^t(U_i)$ and a parameter value σ to the position $\gamma(\sigma) \in U_i$ of the path at that parameter value, and W_{A_i} denotes the holonomy of A_i along γ, from the integration parameter to the endpoint.

So this formula tells us to pull back the 2-form B_i from U_i to $P_s^t(U_i) \times [0,1]$ using the evaluation map, and then to integrate the result over the path γ. The term $\alpha(W_{A_i})$ in this formula indicates that, while doing this integration, we are to use the ordinary line holonomy

$$W_{A_i}(\gamma) = \mathrm{hol}_i(\gamma)$$

to continuously parallel transport $\mathrm{ev}^*(B_i)$ to the endpoint of the path.

But it turns out that not all combinations (A_i, B_i) correspond to holonomy 2-functors $\mathrm{hol}_i\colon \mathcal{P}_2(U_i) \to \mathcal{G}$. Instead, the 1-forms A_i and 2-forms B_i that correspond to holonomy 2-functors satisfy the relation

$$F_{A_i} + dt(B_i) = 0\,.$$

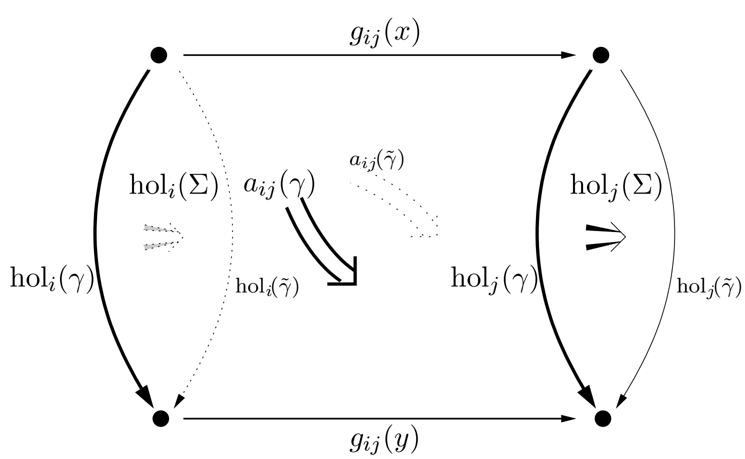
Following [49] we say that the **fake curvature** has to vanish. This relation can be shown to encode the functoriality of hol_i, i.e. the fact that hol_i respects the combined horizontal and vertical composition of surfaces in $\mathcal{P}_2(U_i)$.

While a gauge transformation for an ordinary (1-)holonomy is the same as a natural isomorphism between (1-)functors, a gauge transformation for the above holonomy 2-functors is a pseudo-natural isomorphism (*cf.* §4.3.2.1 (p.81)).

This means that for every surface $\gamma \xrightarrow{\Sigma} \tilde{\gamma} \in \mathrm{Mor}_2(\mathcal{P}_2(U_{ij}))$ in a double overlap, there are 2-group elements

$$a_{ij}(\gamma)\,, a_{ij}(\tilde{\gamma})$$

such that the images of Σ under hol_i and hol_j are related by the following 2-commuting diagram:



We will show that in terms of the differential forms (A_i, B_i) that determine the 2-functors hol_i, the 2-commutativity of this diagram implies that there exist 1-forms

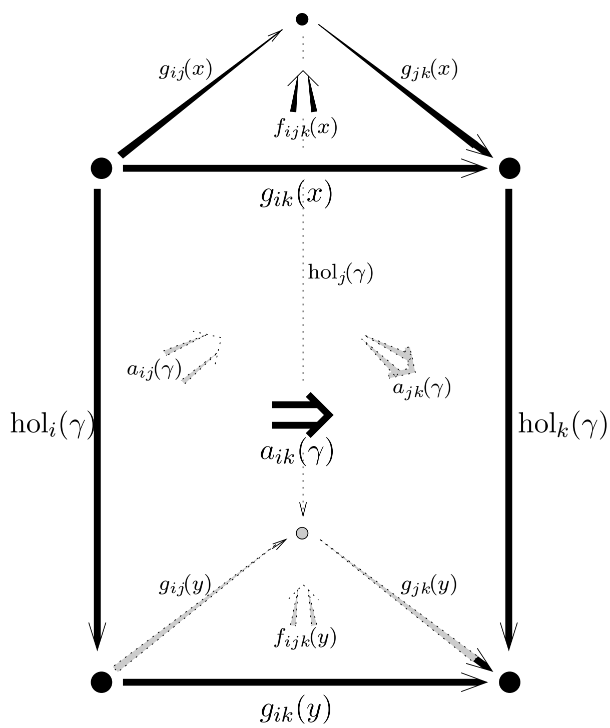
$$a_{ij} \in \Omega^1(U_{ij}, \mathfrak{h})$$

such that the following equations hold

$$\begin{aligned} A_i &= g_{ij} A_j g_{ij}^{-1} + g_{ij} \mathbf{d} g_{ij}^{-1} - dt(a_{ij}) \\ B_i &= \alpha(g_{ij})(B_j) + \mathbf{d} a_{ij} + a_{ij} \wedge a_{ij} \,. \end{aligned}$$

These are the transition laws (cocycle relations) for A_i and B_i.

For this to be consistent, there is a condition on the a_{ij}. This is expressed by the requirement that on triple overlaps U_{ijk} the following diagram 2-commutes, expressing the existence of a 2-morphism of 2-functors (a modification):



It can be shown that this is the case when in terms of local data the following equation holds:

$$f_{ijk} d\alpha(A_i)\left(f_{ijk}^{-1}\right) + f_{ijk} \mathbf{d} f_{ijk}^{-1} - a_{ij} - g_{ij}^{1}(a_{jk}) + f_{ijk} a_{ik} f_{ijk}^{-1} = 0 \,.$$

This is the cocycle condition on the a_{ij}.

These cocycle relations have previously been found by other methods in the context of nonabelian gerbes in [49, 50]. Here we obtain them for the special case that the fake curvature vanishes. The notion of 2-connection for nonabelian gerbes should come from categorifying a different definition of ordinary connections than we used here, one that

does not refer to holonomy. The vanishing of the fake curvature, as we have mentioned, is a result of our insistence on having a notion of 2-holonomy.

It can now be shown that the local 2-holonomy functors hol_i can be glued together to a global functor that computes global 2-holonomy.

In generalization of the situation for ordinary bundles, this requires covering the surface with patches U_i and picking a trivalent graph on the surface such that each face comes to sit in a single patch, each edge in a double overlap and each vertex in a triple overlap. This procedure is familiar from the theory of surface holonomy for abelian gerbes [42, 48, 45]. Then to each face of the graph we can associate the surface holonomy computed by the local holonomy 2-functor, and the resulting 2-group elements are glued along their common boundaries by means of f_{ijk} and a_{ij}, which generalize the transition function g_{ij} known from ordinary bundles. The resulting action of the global 2-holonomy 2-functor is illustrated by figure 8 (p. 24).

While this may look complicated at first sight, we would like to emphasize the direct analogy of this procedure to the one for ordinary 1-connections in 1-bundles, described in §3.3.1 (p.55). In fact, staring at figure 8 for a moment reveals that it simply says that the local surface holonomies hol_i have to be glued in the only possible way using the 2-group elements f_{ijk} and a_{ij}, just like global 1-holonomy was obtained by gluing local 1-holonomies in the only possible way using the group element g_{ij}.

In the context of abelian gerbes with connection and curving a formula for how to compute a global surface holonomy is well known [42, 48, 45]. It is easily seen that this formula arises from the above diagrammatic prescription in the special case where the structure 2-group G_p comes from a crossed module of the form $G_2 = (1, H, \alpha = \text{trivial}, t = \text{trivial})$, with H an abelian Lie group. The above gives a diagrammatic understanding of this formula and generalizes it to more general nonabelian strict 2-groups.

In fact, all the diagrams that we display essentially apply directly to the much more general case where G_p is any weak coherent 2-group or even just a coherent 2-groupoid. The minor refinement necessary to describe the weak case is discussed in an example in §11.2.5 (p.252). What is however much harder for non-strict structure 2-groups is to find the expression of the respective 2-holonomies and their transition relations in terms of local differential forms. These are much more conveniently found using the differential formulation of p-bundles with p-connections that is discussed in §3.4 below.

3.4 The Differential Picture: Nonabelian Deligne Hypercohomology

The above discussion focused on and was motivated by the desire to write down global surface holonomies, i.e. to associate p-group elements to p-paths. As section §11.5 (p.268) will demonstrate, there is quite some gymnastics in path-space differential geometry required when these "integral" notions are to be translated into local differential forms, like A_i and B_i.

However, there are situations in which one will be more interested in these local differential forms than in the 2-holonomy that they give rise to. Most notably, once we are interested not so much in the dynamics of strings in the background of the fields described

by these local forms, but in the dynamics of these background fields themselves, the indirect definition of these fields in terms of p-holonomy p-functors becomes unwieldy.

For instance, the holonomy 1-functor which was discussed in §3.3.1 (p.55) does allow to write down the Yang-Mills action, at least on the lattice, using Wilson's prescription, but in the continuum limit we will want to use the action functional in the form $\int_{U_i} \mathrm{Tr}(F_{A_i} \wedge \star F_{A_i})$, making use of A_i, which is only somewhat indirectly encoded by hol_i.

Similarly constructing interesting and sensible action functionals for the differential forms that appear in 2-bundles with 2-connection is already a rather more delicate issue, and it turns out that the "integral formalism" described above is a clumsy tool for attacking such issues.

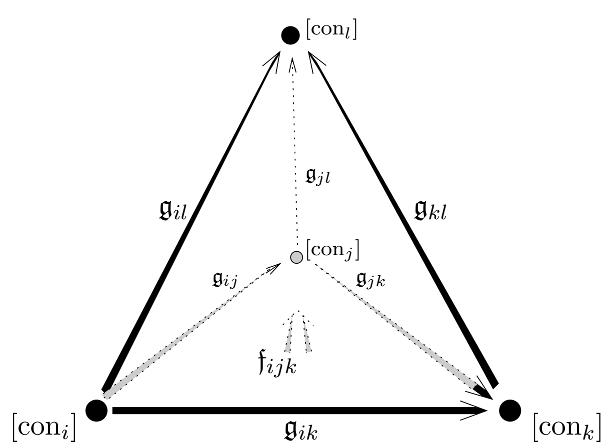
Recently it had been noticed, for instance in [37], that, apparently, using differential graded algebras (dg-algebras) a natural and much more powerful language for dealing with higher nonabelian p-form gauge theories can be obtained. While superficially this approach may look rather unrelated to the considerations presented here, they are in fact closely related, as illustrated in figures 6 (p. 19) and 7 (p. 21).

The point is that, like a Lie group can be "differentiated" to a Lie algebra, there should be a differential analog of the local p-holonomy p-functor hol_i, called a **local p-connection p-morphism**, which associates n-morphisms of a Lie n-algebra to differential n-forms. We write this as

$$\mathrm{con}_i \colon \mathfrak{p}_p(U_i) \to \mathfrak{g}_p \,,$$

where $\mathfrak{p}_p(U_i)$ is an algebroid called the p-path p-algebroid and $\mathfrak{g}_p$ is the Lie p-algebra associated to the structure p-group G_p.

The local p-connection p-functors con_i glue together on multiple overlaps just as before. Hence on double overlaps there is a 1-morphisms of local p-connection p-morphisms, on triple overlaps there is a 2-morphism between such 1-morphisms, and so on.



It is known that these Lie p-algebras, at least as long as they are what is called "semistrict", have a dual description in terms of dg-algebras. In this language the local p-connection becomes a morphism of dg-algebras, which is known as a *chain map*. A 2-morphism between such 1-morphisms is what is known as a *chain homotopy* between chain maps, and so on.

This has the following interesting consequence:

There is a natural nilpotent operator Q acting on the space of dg-algebra n-morphisms, which is essentially the commutator with the differentials of the target and the source dg-algebra. Two $(n-1)$-morphisms of dg-algebras are related by an n-morphisms of dg-algebras precisely if they differ by a Q-exact term.

Moreover, there is another nilpotent operator, δ, which sends an n-morphism to the linear combination of its source and target $(n-1)$-morphisms, which label (recall the discussion in §3.3 (p.52)) the faces of an n-simplex. More precisely, this δ is nothing but the *Čech coboundary operator* on the complex of sheaves of dg-algebra n-morphisms.

The point is that the differential characterization of a p-bundle with p-connection, which, according to §3.3 (p.52), is the assignment, ω, of dg-algebra n-morphisms to n-faces of a p-simplex, is concisely encoded in the equation

$$(\delta + Q)\omega = 0\,.$$

When the details are unraveled, this equation says nothing but that the $(n-1)$-morphisms that ω associates to the $(n-1)$-faces of an n-simplex are the source and target of the n-morphism associated to the simplex itself.

In other words, every ω in the kernel of the nilpotent operator

$$D = \delta + Q$$

specifies the differential version of the local trivialization of a p-bundle with p-connection.

Moreover, a gauge transformation in that local trivialization corresponds to shifting ω by a D-exact term

$$\omega \to \omega + D\lambda\,.$$

This way gauge equivalence classes of the differential version of p-bundles with p-connection can be characterized by *cohomology classes* of the operator D.

In the special case that the target Lie p-algebra $\mathfrak{g}_p$ is abelian and strict, one finds that the above operator D reduces to what is known as the **Deligne coboundary operator**. The cohomology of this operator is well known to describe abelian gerbes with connection and curving and hence abelian 2-bundles with 2-connection. Our generalized operator D should hence be called a **nonabelian Deligne coboundary operator**. A better term might be **generalized Deligne coboundary operator**, since general $\mathfrak{g}_p$ may differ from strict abelian Lie p-algebras not only in being non-abelian, but also in being semistrict or being p-algebroids instead of p-algebras.

However, the nonabelian Deligne coboundary operator captures terms in addition to those seen by the ordinary abelian Deligne operator only to linear order. It lives in a fiber to the tangent bundle to the space of all p-connections instead of on all of that space. Hence, in general, a cohomology class of the generalized nonabelian Deligne coboundary operator does not classify an "integral" p-bundle with p-connection.

We will demonstrate in §13.5 (p.355) and §13.6 (p.359), for the cases where $\mathfrak{g}_p$ is the differential version of a strict 1-group or of a strict 2-group G_p, how the equation $D\omega = 0$ does indeed encode the differential (linearized) version of all the cocycle relations that

were discussed in §3.3.1 (p.55) and §3.3.2 (p.57), and how a shift $\omega \to \omega + D\lambda$ does indeed describe the differential version of the gauge transformation laws for these cases.

Therefore the differential picture and the integral picture of p-bundles with p-connection are somewhat complementary. While in the differential picture all equations are obtainable only "infinitesimally", it can easily deal with situations that are hard or even impossible to treat using the integral formulation.

For instance, there are non-strict Lie p-algebras that are known not to be integrable to any Lie p-group. One example for these was the family of Lie 2-algebras called $\mathfrak{g}_k$ in §3.1.4 (p.43). As was discussed there, while $\mathfrak{g}_k$ itself is not integrable, it is equivalent, in the category-theoretic sense, to an infinite-dimensional strict Lie 2-algebra which is. The differential formalism discussed above now allows to use $\mathfrak{g}_k$ itself as the "structure 2-algebra" of a differential 2-bundle, and to analyze the classification of these differential 2-bundles by studying the respective generalized Deligne cohomology classes - even though there is no integral 2-bundle directly related to this. The discussion of this in §13.7 (p.365) concludes the investigations to be presented here.

4. More Background and More on Motivations

Much more can be said concerning the motivations, background and related literature of the ideas presented here than was done in §1.1 (p.10). The following is an attempt to give a somewhat more detailed account

- of the literature on membranes attached to 5-branes in §4.1.1 (p.64),
- of n^3-scaling in these theories and how this could be described by 2-bundles with 2-connections in §4.1.2 (p.66),
- of the nature of spinning strings in §4.2 (p.68),
- of the concepts of (n-)category theory in §4.3 (p.72),
- of the possible relations between 2-bundles and the derived category description of open strings on D-branes in §4.4 (p.83).

4.1 Open Membranes on 5-Branes

4.1.1 Literature

The target space theories which give rise to non-abelian 2-forms are not at all well understood [51]. One expects [52, 53] that they involve stacks of 5-branes on which open membranes may end [54, 55, 56]. This has recently been made more precise [23] using anomaly cancellation on M5-branes and the language of nonabelian gerbes developed in [50]. The boundary of these membranes appear as strings, [57, 58], (self-dual strings [59, 60, 61], "little strings" [62], fundamental strings or D-strings [63]) in the world-volume theory of the 5-branes [56], generalizing [51] the way how open string endpoints appear as "quarks" in the world-volume theory of D-branes. Just like a nonabelian 1-form couples to these "quarks", i.e. to the boundary of an open string, a (possibly non-abelian) 2-form should couple [64] to the boundary of an open membrane [58, 65, 66, 67], i.e. a to string on the (stack of) 5 branes. One proposal for how such a non-abelian B field might be induced by a stack of branes has been made in [64]. A more formal derivation of the non-abelian 2-forms arising on stacks of M5 branes is given in [23]. General investigations into the possible nature of such non-abelian 2-forms have been done for instance in [68, 69].

(From the point of view of the effective 6-dimensional supersymmetric worldvolume theory of the 5-branes these 2-form field(s) come either from a tensor multiplet or from a gravitational multiplet of the worldvolume supersymmetry representation [70].)

This analogy suggests that there is a *single* Chan-Paton-like factor associated to each string living on the stack of 5 branes, indicating which of the N branes in the stack it is associated with. This Chan-Paton factor should be the degree of freedom that the non-abelian B-field acts on.

Hence the higher-dimensional generalization of ordinary gauge theory should, in terms of strings, involve the steps upwards the dimensional ladder indicated in table 1.

(1-)gauge theory		2-gauge theory
string ending on D-brane	$\to$	membrane ending on NS brane
"quark" on D-brane	$\to$	string on NS brane
nonabelian 1-form gauge field	$\to$	nonabelian 2-form gauge field B
coupling to the boundary of a 1-brane (string)	$\to$	coupling to the boundary of a 2-brane (membrane)
Chan-Paton factor indicating which D-brane in the stack the "quark" sits on	$\to$	Chan-Paton-like factor indicating wich NS brane in the stack the membrane boundary string sits on.

Table 1: Expected relation between 1-form and 2-form gauge theory in stringy terms

These considerations receive substantiation by the fact that, indeed, the contexts in which nonablian 2-forms have been argued to arise naturally are the worldsheet theories on these NS 5-branes [53, 52, 57, 63, 51, 71].

The study of little strings, tensionless strings and $N = (2, 0)$ QFTs in six dimensions is involved, and no good understanding of any non-abelian 2-form from this target space perspective has emerged so far. However, a compelling connection is the relation of these 6-dimensional theories, upon compactification, to Yang-Mills theory in 4-dimensions, where the 1-form gauge field of the Yang-Mills theory arises as one component of the 2-form in the 6-dimensional theory [71].

For this to work the dimension $d = 5 + 1$ of the world-volume theory of the 5-branes plays a crucial role, because here the 2-form B can have and does have *self-dual* field strength $H = \star H$ [58, 71] (related to the existence of the self-dual strings in 6 dimensions first discussed in [72]).

But this means that there cannot be any ordinary non-topological action of the form $dH \wedge \star dH$ for the B-field, and that furthermore the dynamical content of the B field would essentially be that of a 1-form α [71]: Namely when the 1+5 dimensional field theory is compactified on a circle and B is rewritten as

$$B = B_{ij}\, dx^i \wedge dx^j + \alpha_i dx^i \wedge dx^6 \qquad \text{for } i, j \in \{1, 2, 3, 4, 5\}\,.$$

with $\partial_6 B = 0$, then $dB = \star dB$ implies that in five dimensions B is just dual to α

$$d^{(5)}B = \star^{(5)} d\alpha\,. \tag{4.1}$$

In particular, since the compactified theory should give possibly non-abelian Yang-Mills with α the gauge field [71] it is natural to expect [53] that in the uncompactified theory there must be a non-abelian B field. Since there is no Lagrangian description of the brane's worldvolume theory [73, 52] it is hard to make this explicit. This is one reason why it seems helpful to consider the worldsheet theory of strings propagating in the 6-dimensional brane volume. The non-abelian Yang-Mills theory in the context of NS 5-branes considered in [57] uses n D4-branes suspended between two NS5-branes. The former can however be regarded as a single M5-brane wrapped n times around the S^1 (*cf.* p. 34 of [57]).

In [74] it is argued that, while the worldvolume theory on a stack of 5-branes with non-abelian 2-form fields is not known, it cannot be a local field theory. This harmonizes

with the attempts in [51] to define it in terms of "nonabelian surface equations" which are supposed to generalize the well-known Wilson loop equations of ordinary Yang-Mills theory to Wilson *surfaces*. These Wilson surfaces become ordinary Wilson loops in loop space, and should be closely related to the notion of 2-holonomy presented here.

We should emphasize that it is an open question whether gerbes and 2-bundles are the right language to describe stacks of 5-branes, since their physics is not understood well enough at this point.

4.1.2 n^3-Scaling

Albeit the effective field theories on stacks of 5-branes are not very well understood, it it known that they have a property called **n^3-scaling behaviour** [75, 76, 77, 78, 79, 80].

An ordinary gauge theory gives rise to an entropy which asymptotically scales like the *square* of the rank of the gauge group, i.e. its rank. In stringy language this means that the entropy asymptotically scales with the square of the number of coincident D-branes. This can be directly understood as being due to strings which can stretch between $\sim n^2$ possible pairs of these branes, which again is reflected in the $\sim n^2$ entries of the matrix representing the gauge connection.

It turns out, however, that the entropy of theories describing stacks of M5-branes scales with the *cube* of the number of branes in the stack. Even though the implication of this phenomenon for the conceptual nature of the effective field theory on these branes has remained rather mysterious, there are several ways to understand from the string theory picture how this comes about:

Like open strings stretch between D-branes, there are open membranes stretching between M5-branes. These membranes happen to have a BPS state in which their spatial configuration is that of a 'pair of pants'. Therefore, like open strings stretch between pairs of branes, open BPS membranes can stretch between *triples* of branes [81, 82].

Indeed, in [83] the n^3-scaling of theories on stacks of 5-branes has simply been interpreted as being due to the $\sim n^3$ possible triples of 5-branes between which the 'pair-of-pants' BPS state of the M2-brane can stretch.

In light of the above interpretation of n^2-scaling in ordinary gauge theories this strongly suggests that there should be a generalized form of gauge connection in the effective theories of stacks of 5-branes which, in one way or another, generalize the matrix representing the gauge connection to a cubic array of numbers, i.e. to some tensor of total rank three.

An interesting question is therefore if the formalism of 2-holonomy presented here can capture such a phenomenon. While this is an open question not further studied here, we would like to mention two possibly interesting speculations concerning this point.

1. **Correlators in 2D TFTs.**

   It is well known that topological field theories (TFTs) in two dimensions on triangulated manifolds are in 1-1 correspondence with semisimple associative algebras $\mathcal{A}$ [84]. Let ${C_{ab}}^c$ be the structure constants of such an algebra in a given basis. The partition function of the corresponding TFT on a given surface Σ is computed by choosing a triangulation of Σ with oriented edges, assigning a copy of C to each triangle with

each index associated to one of the edges, raising and lowering these indices with the algebra's Killing metric κ_{ab} according to whether the respective edge is ingoing or outgoing, and then contracting all pairs of indices belonging to the same edge.

The associativity of the algebra as well as the non-degeneracy of κ can be seen to ensure that the number obtained this way is independent of the triangulation chosen. It is a topological invariant of Σ.

One can regard any rank 3 tensor $V \in \otimes^3 \mathcal{A}$ as a "vertex operator" for such a theory. The N-point function

$$\langle V_1 V_2 \dots V_N \rangle_\Sigma$$

can be defined by picking a triangulation of Σ with n non-adjacent triangles removed, assigning copies of C to this triangulation as before and assigning the V_i to the triangles that have been removed, contracting all indices as before.

In particular, when we have a 2-form B on Σ which takes values in $\otimes^3 \mathcal{A}$, we can form the correlator

$$h_\Sigma(B) \equiv \left\langle \exp\left(\int_\Sigma B \right) \right\rangle_\Sigma ,$$

which is naturally interpreted as a form of surface holonomy of B over Σ. Essentially this construction has been proposed in [85].

Note how this can be regarded as a direct generalization of a similar formulation of ordinary line holonomy. We could define a trivial 1-dimensional TFT on the lattice by picking some vector space $\mathcal{A}$, assigning the identity operator on $\mathcal{A}$ to intervals of a segmentation of a 1-dimensional manifold, letting vertices be rank-2 tensors in $\mathcal{A}$ and letting the gauge connection A be a vertex-valued 1-form. Then ordinary line holonomy could be written as

$$\left\langle \mathrm{P}\exp\left(\int A \right) \right\rangle$$

with the correlator taking care of the index contraction.

There should be an abelian 2-group describing the above concept of surface holonomy, and the formalism described in part III should allow to get a globally defined notion of surface holonomy of the above kind for the general situation where the 2-form B is only locally defined.

Even though this setup is "abelian", it does have the interesting property that the degrees of freedom encoded in the $\bigotimes^3 \mathcal{A}$-valued 2-form B scale with the *cube* of the dimension of the vector space associated with the algebra $\mathcal{A}$, corresponding to the fact that a simplex in two dimensions has three 1-faces (edges).

2. **Algebroid YM Theories.**

As we have remarked before, the definition of p-bundles with p-holonomy as described in part III (and sketched in figure 7, p. 21) is directly applicable to cases where the

"structure p-group" is really a p-groupoid (see §4.3.1 (p.72) for more on groupoids). In the differential formulation (*cf.* §13 (p.335)) this corresponds to replacing the structure p-algebra by a p-algebroid. Where an ordinary algebra has structure *constants* an algebroid has position-dependent structure *functions*, in a sense. These are again potential candidates for causing n^3-scaling behaviour.

Such structure functions were apparently first considered in the context of higher gauge theory in [53], which is reviewed in §8.3 (p.173). The somewhat more systematic treatment using algebroids was more recently discussed in [37], where some interesting consequences are reported that might be of relevance for nonabelian strings.

In the present context these hints are all that we are going to say about the issue of n^3-scaling in higher gauge theory.

4.2 Spinning Strings

In order to approach the issue of spinning strings, first recall the situation for $\mathrm{Spin}(n)$. A (Riemannian) manifold M *is spin* or *admits a spin structure* if spinning *particles* can consistently propagate on it.

This is the case iff an $\mathrm{SO}(n)$-bundle

$$\begin{array}{c} E \\ \downarrow \\ M \end{array}$$

over the manifold M can be lifted to a $\mathrm{Spin}(n)$-bundle, where $\mathrm{Spin}(n)$ is the central extension of $\mathrm{SO}(n)$ by $\mathbb{Z}/2$:

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(n) \to \mathrm{SO}(n) \to 1\,.$$

This is the case iff M is orientable and the second Stiefel-Whitney class $w_2(E) \in H^1(M;\mathbb{Z}/2)$ vanishes.

The situation for $\mathrm{String}(n)$ is similar, but with everything lifted by one dimension. A manifold *is string* or *admits a string structure* if spinning *strings* can consistently propagate on it.

This is the case iff a principal loop-group $L\mathrm{SO}(n)$-bundle

$$\begin{array}{c} LE \\ \downarrow \\ LM \end{array}$$

over the free loop space LM can be lifted to a $\widehat{L\mathrm{SO}(n)}$-bundle, where $\widehat{L\mathrm{SO}(n)}$ is a (Kac-Moody-)central extension of $L\mathrm{SO}(n)$ by $U(1)$:

$$1 \to U(1) \to \widehat{L\mathrm{SO}(n)} \to L\mathrm{SO}(n) \to 1\,.$$

And this is the case iff the so-called *string class* of LM in $H^3(LM;\,\mathbb{Z})$ vanishes.

These two conditions on the topology of LM can equivalently be formulated in terms of M itself:

1. The vanishing of the string class in $H^3(LM;\mathbb{Z})$ is equivalent to the vanishing of the first Pontryagin class $\frac{1}{2}p_1(E)$ of a vector bundle associated to a principal Spin(n)-bundle $E \to M$.

   The string class in $H^3(LM;\mathbb{Z})$ is obtained from the Pontryagin class $p_1/2$ by *transgression.* This means that it is represented by the 3-form

   $$\int_\gamma \mathrm{ev}^*(\xi)\,,$$

   where ξ is a representative of $p_1/2$, ev^* is the pull-back by the evaluation map

   $$\begin{aligned} \mathrm{ev} : LM \times S^1 &\to M \\ (\gamma,\sigma) &\mapsto \gamma(\sigma) \end{aligned}$$

   and $\int_\gamma$ denotes the integral over the S^1-factor in $LM \times S^1$.

2. This again is equivalent to the existence of a lift of the structure group of E from Spin(n) to the topological group called String(n).

   The group String(n) (or rather a 'realization' thereof) is defined as a topological group all of whose homotopy groups equal those of Spin(n), except for the third one, which has to vanish for String(n):

   $$\pi_k(\mathrm{String}(n)) = \begin{cases} \pi_k(\mathrm{Spin}(n)) & \text{for } k \neq 3 \\ 1 & \text{for } k = 3 \end{cases}$$

   This should be seen in the following context:

   As is well known, the first homotopy groups π_k of the orthogonal group $O(n)$ for $n > 8$ are given by the following table:

k	0	1	2	3	4	5	6	7
$\pi_k(O(n))$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

   The 0-th homotopy group $\pi_0 = \mathbb{Z}/2$ indicates that $O(n)$ is not connected but has two connected components. One can 'kill' this homotopy group by going over to the connected component of the identity element, i.e. to the special orthogonal group $SO(n)$, which has $\pi_0(SO(n)) \simeq 0$.

   The first homotopy group $\pi_1 = \mathbb{Z}/2$ indicates that $O(n)$ is not simply connected. One can 'kill' this homotopy group by going over to its universal double cover, the group Spin(n), which has $\pi_1(\mathrm{Spin}(n)) \simeq 0$.

   All of $O(n)$, $SO(n)$ and Spin(n) are of course semisimple Lie groups. Every semisimple Lie group has nonvanishing π_3. Hence, if one wishes to continue with 'killing' homotopy groups of $O(n)$ this way, one will end up with a group that is no longer smooth. Instead its group space will just be a topological space with the group operation being a continuous map on this space. Such groups are called topological groups.

It can be shown and is well known that an equivalent way to define (a realization of) the group $\mathrm{String}(n)$ is as the topological group which makes this sequence of groups exact:

$$1 \to K(\mathbb{Z},2) \to \mathrm{String}(n) \to \mathrm{Spin}(n) \to 1\,.$$

Here $K(\mathbb{Z},2)$ denotes (a realization of) the Eilenberg-MacLane space $K(\mathbb{Z},2)$, which is by definition a topological space all whose homotopy groups vanish, except for the second one, which is isomorphic to $\mathbb{Z}$. In general

$$\pi_k(K(G,n)) \simeq \begin{cases} G \text{ for } k = n \\ 1 \text{ otherwise} \end{cases},$$

by definition.

The importance of string structures in string theory results from the fact that superstrings are nothing but "spinning strings", i.e. fermions on loop space, and that their quantum equations of motion are nothing but a generalized Dirac equation on loop space. (The 0-mode of the worldsheet supercharge is a generalized Dirac(-Ramond) operator on loop space (for the closed string).)

It hence follows by the above discussion that superstrings can propagate consistently only on manifolds which *are string*, just like an ordinary point-like fermion can propagate consistently only on a manifold that is *spin*.

More technically, the wavefunction of a point-like fermion is really a section of a $\widehat{\mathrm{SO}(n)} \simeq \mathrm{Spin}(n)$-bundle and hence such a bundle needs to exist over spacetime in order for the fermion to exists.

Similarly, the wavefunction of a fermionic string (spinning string) is really a section of a $\widehat{L\mathrm{SO}(n)}$-bundle over loop space, and hence such a bundle needs to exist over the loop space over spacetime for fermionic strings to exist.

For instance the worldsheet supercharge of the heterotic string is a Dirac operator on loop space for fermions that are also "charged" under an SO(32)- or $E_8 \times E_8$-bundle

$$\begin{array}{c} V \\ \downarrow \\ M \end{array}\,.$$

In K-theory one can form the difference bundle

$$E = V - T\,,$$

where T is the tangent bundle and the condition for this bundle to admit a string structure is that the Pontryagin class vanishes, i.e. that

$$p_2(V) - p_1(T) = 0\,.$$

This is in fact the relation which follows from the cancellation of the perturbative anomaly of the effective SO(32)- or $E_8 \times E_8$-field theory obtained from these strings. Hence this famous anomaly is related to the fact that heterotic strings are spinors on loop space.

One of the earliest discussions of these issues is given in [86]. Killingback discusses the spinning point particle, the spinning string, the obstructions to lifting the $L\mathrm{SO}(n)$-bundle on loop space to the central extension and the relation to the perturbative anomalies of the effective field theory of the heterotic string.

As a supplement to this there is the nice and more detailed discussion of the relation between $H^3(LM;\mathbb{Z})$ and $p_1/2$ as given in [87].

A more detailed discussion of the nature of Dirac operators on loop space with a review of Killinback's results is given in [18] which has the companion paper [88]

Concerning the group $\mathrm{String}(n)$ the best available reference is probably [20].

A discussion of the group $\mathrm{String}(n)$ and of string structures is given on the top of p. 5 of [20] and then in the beginning of section 5 of [20] on pp. 65. The "killing" of homotopy groups is discussed on p. 65 of [20], the definition of $\mathrm{String}(n)$ by means of an exact sequence is discussed on p. 66, and the relation to the Pontryagin class is discussed on p. 67.

4.3 Category Theory

For the physicist's convenience the following gives a quick introduction to some elementary concepts of category theory that are used in the main text.

Literature. The standard introductory textbook for category theory is [89], of which mainly only the first few pages are needed here. The history of n-categorical physics is treated in [90], which also serves as a nice introduction to the elementary concepts. A pedagogical discussion of categorification is given in [25]. Details on 2-categorical technology in the context of categorified gauge theory are given in [91, 92].

4.3.1 Categories

The concept **category** is a generalization of the concept **set**. A category C is a set $\mathrm{Ob}(C) = \{a, b, \ldots\}$, called the set of **objects**, together with a set $\mathrm{Mor}(C) = \{r, s, \ldots\}$ of arrows, called the set of **morphisms**. A morphism is is something that goes between two given objects

$$\begin{tikzcd} a \arrow[r, bend left, "r"] & b \end{tikzcd} .$$

Morphisms can be anything. In particular, they need not be functions or maps between sets. But they can. All one requires is that given two morphisms

$$\begin{tikzcd} a \arrow[r, bend left, "r"] & b \end{tikzcd}$$

and

$$\begin{tikzcd} b \arrow[r, bend left, "s"] & c \end{tikzcd}$$

which are **composable**, i.e. where the target object b of one matches the source object of the other, there is a uniquely defined morphism

$$\begin{tikzcd} a \arrow[r, bend left, "r \circ s"] & c \end{tikzcd} \equiv \begin{tikzcd} a \arrow[r, bend left, "r"] & b \arrow[r, bend left, "s"] & c \end{tikzcd}$$

obtained by **composition**. Furthermore, this composition $\circ$ is supposed to be associative

$$\begin{tikzcd} a \arrow[r, bend left, "(r \circ s) \circ t"] & d \end{tikzcd} \equiv \begin{tikzcd} a \arrow[r, bend left, "r \circ (s \circ t)"] & d \end{tikzcd} \equiv \begin{tikzcd} a \arrow[r, bend left, "r"] & b \arrow[r, bend left, "s"] & c \arrow[r, bend left, "t"] & d \end{tikzcd} ,$$

and for every object there must be an **identity morphism**

$$\begin{tikzcd} a \arrow[r, bend left, "\mathrm{Id}"] & a \end{tikzcd}$$

such that its composition with any other morphism equals that other morphism:

$$
\begin{aligned}
& a \xrightarrow{\mathrm{Id}} a \xrightarrow{r} b \\
= \; & a \xrightarrow{r} b \xrightarrow{\mathrm{Id}} b \\
= \; & a \xrightarrow{r} b \,.
\end{aligned}
$$

If a morphism

$$a \xrightarrow{r} b$$

is invertible, i.e. if there is another morphism

$$b \xrightarrow{r^{-1}} a$$

such that

$$a \xrightarrow{r} b \xrightarrow{r^{-1}} a \;=\; a \xrightarrow{\mathrm{Id}} a$$

and

$$b \xrightarrow{r^{-1}} a \xrightarrow{r} b \;=\; b \xrightarrow{\mathrm{Id}} b \,,$$

then r is called an **isomorphism**.

4.3.1.1 Examples. One large class of categories that everybody is familiar with consists of categories whose objects are sets with a given extra structure and whose morphisms are maps between sets that preserve this structure.

So there is the category

- Set, whose objects are all (small) sets and whose morphisms are maps between these sets.
- Top, whose objects are topological spaces (i.e. sets equipped with a topology) and whose morphisms are continuous maps between topological spaces.
- Diff, whose objects are smooth spaces (i.e. sets equipped with a smooth stucture) and whose morphisms are smooth maps between such spaces.
- Vect, whose objects are (finite dimensional) vector spaces and whose morphisms are linear maps between these.

In all these cases, the composition of morphisms is nothing but the composition of the respective maps that they represent.

Other categories are rather different in character. For instance consider any oriented graph (V, E), with V a set of vertices and E a set of oriented edges between pairs of vertices.

The free category over (V,E), called $C_{(V,E)}$ is the category whose objects are the vertices in V and whose morphisms are all the paths of edges that one obtains by concatenating edges in V. Composition of morphisms here is concatenation of edge paths.

We may want to think of a graph (V,E) as an approximation to a path space. Let M be any smooth space and let $P(M)$ be the free path space over M, defined to be the space of all smooth maps

$$\gamma \colon [0,1] \to M\,.$$

Here M is like the continuum version of E, while $P(M)$ is like a continuum version of V. In order to get a category out of this, by interpreting a path γ as a morphism between the source object $\gamma(0)$ and the target object $\gamma(1)$, we need to define what the composition of two such paths γ_1 and γ_2 is supposed to be. Of course we want this to be the path obtained by first tracing out γ_1 and then tracing out γ_2. This can be made into an associative operation by going over to thin homotopy equivalence classes of paths, which implies that we forget about the parameterization of these paths. The resulting category is called the **path groupoid** $\mathcal{P}_1(M)$.

Other types of categories have morphisms which are "arrows" in a much more abstact sense. For instance, given any set $\{a,b,\dots\}$ with a partial ordering $\cdot \geq \cdot$, we obtain a category whose objects are elements of this set and which has precisely one morphism

$$\begin{tikzcd} a \arrow[r, bend left] & b \end{tikzcd}$$

from element a to element b precisely if $a \geq b$ with respect to the partial ordering.

The example of relevance in our context is the category $\mathcal{O}(M)$ of open subsets of a topological space M with the partial ordering that given by inclusion $U_1 \supset U_2$ of subset U_2 in subset U_1.

A related example is the category called the **Čech-groupoid**. Given any smooth space M with good covering $\mathcal{U} = \bigsqcup_{i \in I} U_i$ (i.e. a covering of M by open sets such that every finite intersection of these is contractible), its Čech groupoid is the category whose objects are all pairs

$$(x,i)$$

with $x \in U_i$, and which has precisely one morphism

$$\begin{tikzcd} (x,i) \arrow[r, bend left] & (x,j) \end{tikzcd}$$

whenever x is a point in the double overlap $U_{ij} = U_i \cap U_j$.

This category is called a groupoid because all of its morphisms are invertible.

A **groupoid** is a category all whose morphisms are invertible, i.e. all whose morphisms are isomorphisms.

Since, by definition of the Čech-groupoid, there is a unique morphism between any two such pairs and because, by definition of a category, there must be an identity morphism from each object to itself, it follows that the composition of

$$\begin{tikzcd} (x,i) \arrow[r, bend left] & (x,j) \end{tikzcd}$$

with

$$\begin{tikzcd}(x,j) \arrow[r, bend left] & (x,i)\end{tikzcd}$$

equals the identity morphism

$$\begin{tikzcd}(x,i) \arrow[r, bend left, "\mathrm{Id}"] & (x,i)\end{tikzcd} \; .$$

The reason for the term "groupoid" is that in the case that such a category has just a single object, it is the same as an ordinary group:

A **group** is a category with just a single object and all morphisms invertible.

So a group is a groupoid with a single object.

Namely with a single object, $\bullet$, every morphism

$$\begin{tikzcd}\bullet \arrow[r, bend left, "g"] & \bullet\end{tikzcd}$$

can be composed with any other

$$\begin{tikzcd}\bullet \arrow[r, bend left, "g'"] & \bullet\end{tikzcd}$$

(since their source and target always match). Hence composition of morphisms

$$\begin{tikzcd}\bullet \arrow[r, bend left, "g"] & \bullet \arrow[r, bend left, "g'"] & \bullet\end{tikzcd}$$

becomes an associative product map

$$g \circ g' \equiv gg'$$

from the set of morphisms to itself. Since, by assumption, to every morphism there is an inverse morphism, this product operation is that of a group.

For illustration purposes, if one likes, one can think of the morphisms here again as maps, but that's not compulsory. So for instance given the permutation group S_p, one might want to identitfy the single object $\bullet$ with any given p-element set

$$\bullet = \{1, 2, 3, \ldots p\}$$

and identify any element $g \in S_p$ with the map

$$\begin{tikzcd}\{1, 2, \ldots p\} \arrow[r, bend left, "g"] & \{1, 2, \ldots p\}\end{tikzcd}$$

that permutes these p elements.

4.3.1.2 Functors. Categories consist of objects and morphisms. A map from one category to another should respect the composition of these morphisms. In analogy to how a map between sets is called a function, such a map between categories is called a **functor**.

So given categories C and D, a functor F

$$\begin{tikzcd} C \arrow[r, bend left, "F"] & D \end{tikzcd}$$

is a map from the set of morphisms of C to the set of morphisms of D such that

$$F\left(\begin{tikzcd} a \arrow[r, bend left, "r"] & b \end{tikzcd} \right) \circ F\left(\begin{tikzcd} b \arrow[r, bend left, "s"] & c \end{tikzcd} \right) = F\left(\begin{tikzcd} a \arrow[r, bend left, "r \circ s"] & c \end{tikzcd} \right) .$$

For instance, a functor from a group G to a group H (regarded categories with single objects and all morphisms invertible) is nothing but a group homomorphism $G \to H$.

Or consider a functor

$$F \colon C_{(V,E)} \to G$$

from the free category $C_{(V,E)}$ of a graph (V, E) to any group G. Such a functor sends all vertices in the graph to the unique object $\bullet$ in G and labels each edge path of the graph, say $x \xrightarrow{\gamma_1} y \xrightarrow{\gamma_2} z$, with a given group element in G, such that concatenation of edge paths corresponds to multiplication of their group elements:

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g"] & \bullet \arrow[r, bend left, "g'"] & \bullet \\ x \arrow[r, bend left, "\gamma_1"] & y \arrow[u, Rightarrow, "F"'] \arrow[r, bend left, "\gamma_2"] & z \end{tikzcd}$$

This is nothing but what happens in ordinary local holonomy in lattice gauge theory. Paths are consistently labeled by group elements.

Another example relevant for gauge theory is that of a pre-sheaf. A pre-sheaf over a topological space M is the assignment of some set $S(U)$ to each open subset $U \subset M$, together with a restriction map $S(U_1) \to S(U_2)$ whenever $U_1 \supset U_2$, such that restricting first from U_1 to U_2 and then to U_3 is the same as restricting directly to U_3. This can be summarized by saying that a pre-sheaf is a functor

$$S \colon \mathcal{O}(M) \to \mathrm{Set}$$

from the category of open subsets of M to the category of all (small) sets.

For instance, if one lets $S(U)$ be the set of all continuous complex-valued functions over U and lets the restriction map be the restriction of the domain of these functions, one obtains a pre-sheaf (and in fact a sheaf).

4.3.1.3 Natural Transformations. One should think of the functor $F\colon C \to D$ as something that produces an image of any diagram in C in terms of a diagram in D. While the images of two functions can be either equal or not, the images of two functors, being "1-dimensional" in a sense, can be "congruent" or "homotopic" without being equal.

Given two functors

$$F_1\colon S \to T$$

and

$$F_2\colon S \to T$$

one says that there is a **natural transformation**

$$F_1 \xrightarrow{\eta} F_2$$

from F_1 to F_2, if for every object a in the source category S there is morphism $\eta(a)$ in the target category T, such that all diagrams of the following kind commute in T:

$$\begin{array}{c|c} S & T \\ \hline \begin{tikzcd} a \arrow[d, "r"'] \\ b \end{tikzcd} & \begin{tikzcd} F_1(a) \arrow[r, "\eta(a)"] \arrow[d, "F_1(r)"'] & F_2(a) \arrow[d, "F_2(r)"] \\ F_1(b) \arrow[r, "\eta(b)"'] & F_2(b) \end{tikzcd} \end{array}$$

In other words, there is a natural transformation $F_1 \xrightarrow{\eta} F_2$ if the images of any morphism

$$\begin{tikzcd} a \arrow[r, bend left, "r"] & b \end{tikzcd}$$

under $F_{1,2}$

$$F_{1,2}\left(\begin{tikzcd} a \arrow[r, bend left, "r"] & b \end{tikzcd} \right) \equiv \begin{tikzcd} F_{1,2}(a) \arrow[r, bend left, "F_{1,2}(r)"] & F_{1,2}(b) \end{tikzcd}$$

are related by

$$\begin{tikzcd} F_1(a) \arrow[r, bend left, "\eta(a)"] & F_2(a) \arrow[r, bend left, "F_2(r)"] & F_2(b) \end{tikzcd} = \begin{tikzcd} F_1(a) \arrow[r, bend left, "F_1(r)"] & F_1(b) \arrow[r, bend left, "\eta(b)"] & F_2(b) \end{tikzcd} .$$

One should think of η as being a "translation" of F_1 to F_2 using the "paths" in T. We can also write the natural transformation η like this:

$$\begin{tikzcd} S \arrow[r, bend left, "F_1"{name=U}] \arrow[r, bend right, "F_2"'{name=D}] & T \arrow[Rightarrow, from=U, to=D, "\eta"'] \end{tikzcd}$$

If all morphisms $\eta(a)$ are isomorphisms, then η itself is invertible as a natural transformation and is called a **natural isomorphism**.

For instance recall the functor $F\colon C_{(V,E)} \to G$ from the free category of a graph (V,E) to any group G, which we noticed can be interepreted as assigning to each path of edges (morphism in $C_{(V,E)}$) a holonomy (morphism in G). A natural transformation of such a functor is the assignment of group elements $\eta(x)$ to every vertex $x \in V$, such that all diagrams of the following form commute:

$C_{(V,E)}$	G
$\begin{tikzcd} x \arrow[d, "\gamma"'] \\ y \end{tikzcd}$	$\begin{tikzcd} \bullet \arrow[r, "\eta(x)"] \arrow[d, "F_1(\gamma)"'] & \bullet \arrow[d, "F_2(\gamma)"] \\ \bullet \arrow[r, "\eta(y)"'] & \bullet \end{tikzcd}$

This implies that

$$\eta(x)\, F_2(\gamma) = F_1(\gamma)\, \eta(y) \ ,$$

where the product is that in the group G. Multiplying with $(\eta(y))^{-1}$ from the right gives

$$F_2(\gamma) = \eta(x)\, F_2(\gamma)\, \eta^{-1}(y) \ .$$

This is nothing but a **gauge transformation** of the holonomy F_1.

Since a natural transformation goes between functors and since composition of natural transformations is associative, these transformations can be regarded as morphism of a category themselves. Given categories C and D we denote by C^D the **functor category** whose objects are functors $D \to C$ and whose morphisms are natural transformations between these functors.

4.3.2 2-Categories

One remarkable thing about natural transformations is that they are morphisms that go between functors – which are morphisms themselves.

There is a category, called Cat, whose objects are all (small) categories and whose morphisms are functors between these categories. But since there are now also natural transformations between these functors, Cat is really a 2-category.

A **2-category** consists of a set of objects

$$a, b, \dots \ ,$$

together with a set of morphisms

$$\begin{tikzcd} a \arrow[r, bend left, "r"] & b \end{tikzcd}$$

between these objects, together with a set of **2-morphisms**

$$\begin{tikzcd} a \arrow[r, bend left, "r"] \arrow[r, bend right, "s"'] \arrow[r, phantom, "\rho \Downarrow"] & b \end{tikzcd}$$

between these morphisms. In what are called **strict 2-categories** the 1-morphisms as well as the 2-morphisms have an associative composition operation. In addition to the obvious "vertical" composition of 2-morphisms

$$\begin{tikzcd} a \arrow[r, bend left=60, "r"] \arrow[r, "s"] \arrow[r, bend right=60, "t"'] \arrow[r, phantom, shift left=3ex, "\Downarrow \rho"] \arrow[r, phantom, shift right=3ex, "\Downarrow \sigma"] & b \end{tikzcd}$$

there is now also a "horizontal" composition

$$\begin{tikzcd} a \arrow[r, bend left, "r"] \arrow[r, bend right, "s"'] \arrow[r, phantom, "\rho \Downarrow"] & b \arrow[r, bend left, "r'"] \arrow[r, bend right, "s'"'] \arrow[r, phantom, "\sigma \Downarrow"] & c \end{tikzcd} .$$

One demands that horizontal and vertical compositon are compatible in that the order in which they are applied in diagrams like

$$\begin{tikzcd} a \arrow[r, bend left=60, "r"] \arrow[r, "s"] \arrow[r, bend right=60, "t"'] \arrow[r, phantom, shift left=3ex, "\rho \Downarrow"] \arrow[r, phantom, shift right=3ex, "\tilde{\rho} \Downarrow"] & b \arrow[r, bend left=60, "r'"] \arrow[r, "s'"] \arrow[r, bend right=60, "t'"'] \arrow[r, phantom, shift left=3ex, "\sigma \Downarrow"] \arrow[r, phantom, shift right=3ex, "\tilde{\sigma} \Downarrow"] & c \end{tikzcd}$$

is immaterial. This is called the **exchange law**.

For example the category called the path groupoid $\mathcal{P}_1(M)$ has a natural extension to a 2-category $\mathcal{P}_2(M)$, called the 2-path 2-groupoid $\mathcal{P}_2(M)$, whose 2-morphisms

$$\begin{tikzcd} x \arrow[r, bend left, "\gamma_1"] \arrow[r, bend right, "\gamma_1"'] \arrow[r, phantom, "\Sigma \Downarrow"] & y \end{tikzcd}$$

are surfaces interpolating between paths γ_1 and γ_2. Horizontal and vertical composition of these 2-morphisms comes from the ordinary horizontal and vertical gluing of these surfaces.

Another example for a 2-category is a 2-group. In §3.1.2 (p.37) we have introduced a 2-group as a 1-category with a group product functor on it. Like an ordinary group can be regarded as a 1-category with a single object, such a 2-group can be regarded as a 2-category with a single object.

The 2-categorical nature of the category Cat of all (small) categories is what gives rise to the notion of **equivalence of categories** that in the present context plays an important role in the study of the 2-group $\mathcal{P}_kG$ in §3.1.4 (p.43) and §10 (p.200).

Namely, in the presence of 2-morphisms it becomes unnatural to talk about strict invertibility of 1-morphisms. In a 1-category, a (1-)morphism r is invertible (is an isomorphism) if there exists another (1-)morphism r^{-1}, such that the (1-)morphism $r \circ r^{-1}$ *equals* the identity 1-morphism. In a 2-category these 1-morphisms are related by 2-morphisms, and hence we do not ask if two 1-morphisms are equal, but if they are *(2-)isomorphic*, i.e. if they are related by an invertible 2-morphism.

So in a 1-category, two objects a and b are isomorphic if there exist 1-morphisms

$$\begin{tikzcd} a \arrow[r, bend left, "r"] & b \end{tikzcd}$$

and

$$\begin{tikzcd} b \arrow[r, bend left, "r^{-1}"] & a \end{tikzcd}$$

such that their composition equals the identity morphism, which means that there is are identity 2-morphisms

$$\begin{tikzcd} a \arrow[r, bend left, "r \circ r^{-1}", ""{name=U, below}] \arrow[r, bend right, "\mathrm{Id}"', ""{name=D, above}] & a \arrow[from=U, to=D, Rightarrow, "="'] \end{tikzcd}$$

and

$$\begin{tikzcd} b \arrow[r, bend left, "r^{-1} \circ r", ""{name=U, below}] \arrow[r, bend right, "\mathrm{Id}"', ""{name=D, above}] & b \arrow[from=U, to=D, Rightarrow, "="'] \end{tikzcd} .$$

Obviously, in a 2-category, which has nontrivial 2-morphisms, this should be generalized to the statement that two objects a and b are *equivalent* if there are 1-morphisms r and $\bar{r}$ together with *invertible* 2-morphisms ξ and χ of the form

$$\begin{tikzcd} a \arrow[r, bend left, "r \circ \bar{r}", ""{name=U, below}] \arrow[r, bend right, "\mathrm{Id}"', ""{name=D, above}] & a \arrow[from=U, to=D, Rightarrow, "\xi"'] \end{tikzcd}$$

and

$$\begin{tikzcd} b \arrow[r, bend left, "\bar{r} \circ r", ""{name=U, below}] \arrow[r, bend right, "\mathrm{Id}"', ""{name=D, above}] & b \arrow[from=U, to=D, Rightarrow, "\chi"'] \end{tikzcd} .$$

Hence the correct notion of “sameness” of categories is equivalence in this sense. Two categories are equivalent, if they are equivalent as objects of the 2-category Cat. This means that they are equivalent if there are functors (1-morphisms in Cat) going between them whose compositions are naturally isomorphic (related by an invertible 2-morphism in Cat) to the identity functor.

4.3.2.1 2-Functors and Pseudo-Natural Transformations. In an obvious generalization of the concept of an ordinary functor, a 2-functor is a map from a 2-category to another 2-category which respects horizontal and vertical composition of 2-morphisms (and hence also ordinary composition of 1-morphisms). The obvious generalization of a natural transformation between two functors is called a **pseudo-natural transformation**, and it again formalizes the idea that the images of two 2-functors are "congruent".

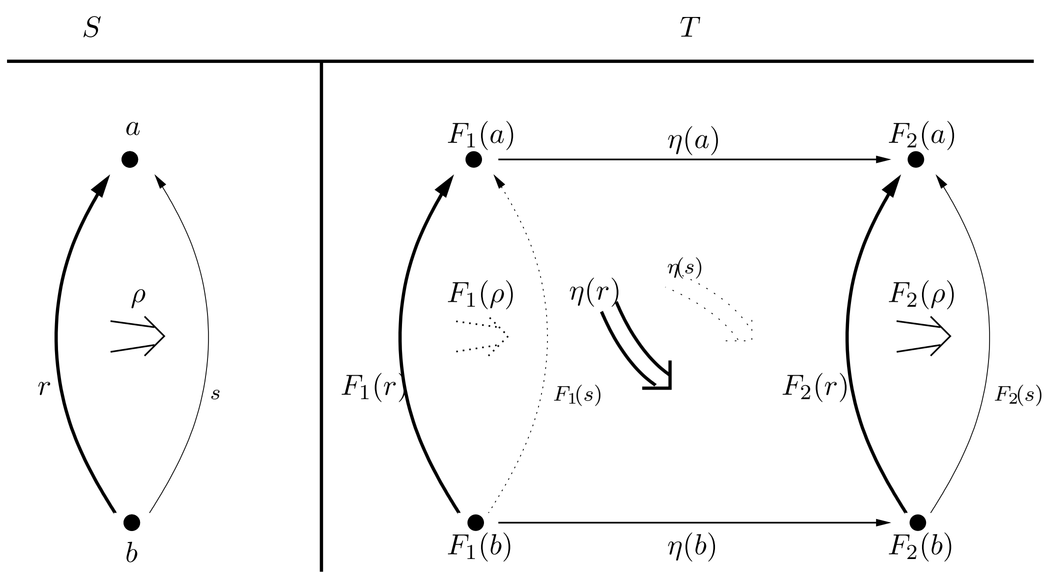
Hence, given 2-functors

$$F_{1,2} \colon S \to T$$

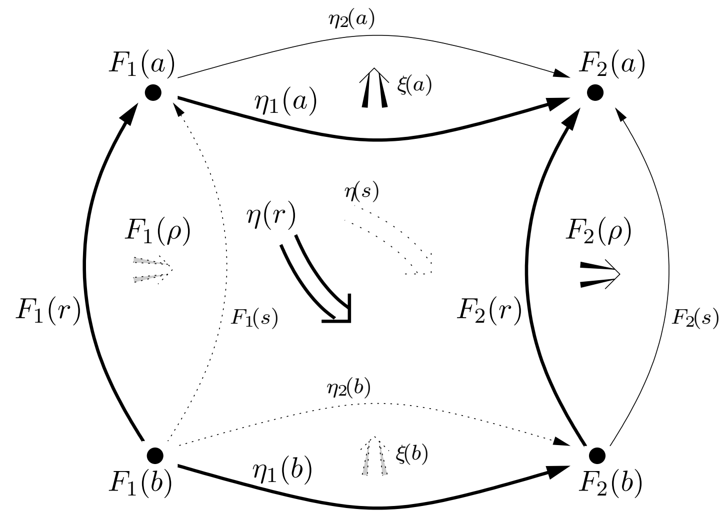
between 2-categories S and T, a pseudo-natural transformation

$$\eta \colon F_1 \to F_2$$

is the assignment of 1-morphisms $\eta(a)$ in T to objects a in S together with the assignment of 2-morphism $\eta(r)$ in T to 1-morphisms r in S, such that diagrams of the following form **2-commute**, meaning that they commute at the level of 2-morphisms:

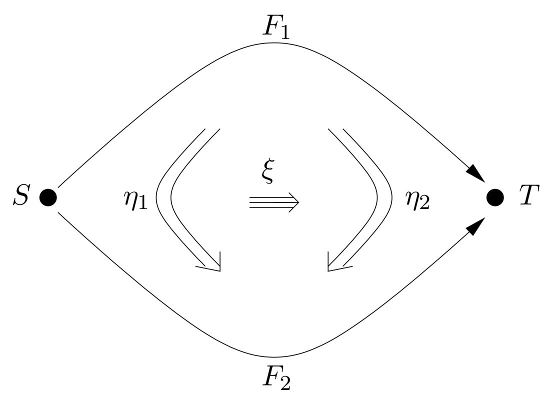


It should not come as a surprise that now we also have transformations between pseudo-natural transformations (3-morphisms). These are called **modifications**. A modification between pseudo-natural transformations η_1 and η_2 is the assignment of a 2-morphism $\xi(a)$ in T to every object a in S, such that these diagrams 2-commute:



(4.2)

Following Baez, we can write such 3-morphisms of 2-functors as



It follows that all 2-functors between given 2-categories C and D form a 2-category C^D, called a **2-functor 2-category**, whose objects are 2-functors $D \to C$, whose morphisms are pseudo-natural transformations η between such functors and whose 2-morphisms are modifications ξ of pseudo-natural transformations.

It should be clear that one can keep on going to n-categories and n-functors for higher and higher n this way. While the basic idea is obvious, each step introduces more and more freedom to “weaken” relations, and the general theory of n-categories is still under development, with the relation between various alternative definitions and approaches remaining to be better understood.

4.4 2-NCG and Derived Category Description of D-Branes

A *field* in physics is something that locally looks like a function on spacetime, but which really is a *section of a bundle*. For instance a spinor field looks like a spinor-valued function locally, but is really, globally, a section of a spinor bundle.

The quanta of ordinary fields are "point particles". Strings, on the other hand, are the quanta of what is called the **string field** (e.g. [17, 93, 94, 95, 96]).

Often, these string fields are treated as nothing but *functions* on space-time that take values in a vector space spanned by the excitation modes of the single first-quantized string. One would expect that, more precisely, globally these string fields should be generalized sections of some generalized notion of fiber bundle, instead.

At least to some extent this is captured by using ordinary fiber bundles on loop space, as we have reviewed in §4.2 (p.68). Given our discussion in §3.1.4 (p.43) and §10 (p.200) on how, in the case of (uncharged) spinors on loop space, this is related to 2-bundles, it is tempting to guess that, more generally, a string field should be a 2-section of some 2-vector 2-bundle.

This is to some degree motivated by our discussion of the relation of the RNS string to supersymmetric quantum mechanics on loop space in §2 (p.28). Given the close relation of SQM to noncommutative spectral geometry (NCG), we can consider states of a supersymmetric particle to be sections of a vector bundle, which arises as a finitely generated projective module of the algebra A of functions on configuration space. It seems to be a plausible conjecture that there is a categorification of this scenario which exhibits string fields as sections of some sort of "2-vector 2-bundle" that arises as a module for a "2-algebra of 2-functions". In fact, aspects of such a setup have been considered in the literature, as discussed below.

In part III we will exclusively deal with principal 2-bundles, since the generalization to the categorification of associated and vector bundles remains to be better understood. But, as a motivation for the general philosophy relating stringification with categorification that emerges from the considerations presented here, and as an outlook for further studies, we would like to sketch in the following some existing approaches and some further observations concerning vector 2-bundles, string fields, categorified supersymmetric quantum mechanics and noncommutative geometry and a possible relation to the description of D-brane states in terms of derived categories.

4.4.1 2-NCG

Ordinary Noncommtative Geometry (NCG) starts with the **Gelfand-Naimark theorem**, which says that a topological space is equivalently encoded in the C^*-algebra of continuous complex-valued functions over it. In the present context we wish to think of such a space as the configuration space of some particle. Upon "stringification" this particle is expected to become a linearly extended entity. Its configurations, when suitably interpreted, include the position of its endpoints together with a specification of how it stretches from one endpoint to the other. The collection of this data, a set of points (objects) and a set of strings (morphisms) between them, may form a category.

Therefore a natural question is whether there is a generalization of the Gelfand-Naimark theorem from sets to categories and if it can serve as a basis for a categorification of all of NCG – and how the result is related to string theory.

The answer to the first part of this question is positive, at least in the case where the underlying spaces are discrete. This, and the idea of categorified Hilbert spaces (which would be the second ingredient in a categorified spectral triple) was discussed in [97].

And indeed, it seems that starting from such a categorified GN theorem and following the logic of categorified NCG one does arrive at descriptions of string physics, as discussed further in §4.4.2 (p.86).

The starting point for turning geometry into algebra is that spaces may be characterized by algebras of functions over them. For instance, topological spaces are characterized by C^*-algebras of continuous functions (the Gelfand-Naimark theorem) and measure spaces by von Neumann algebras of bounded measurable functions.

In each case points of the space X are recovered in terms homomorphisms from the algebra of functions $K^X \equiv \{f : X \to K\}$ to K itself: For every $x \in X$ we get a homomorphism $\tilde{x} \colon K^X \to K$ by setting

$$\begin{aligned} \tilde{x} : K^X &\to K \\ f &\mapsto f(x) \ . \end{aligned}$$

When categorifying, spaces becomes 2-spaces (categories whose point and morphism spaces are topological spaces, or measure space, etc.) and functions become functors.

Let Q be any 2-space and let $\mathcal{K}$ be any monoidal category. The functor category $\mathcal{K}^Q$ (*cf.* §4.3.1.3 (p.77)) now indeed encodes not just the point space of Q, but also the arrow space:

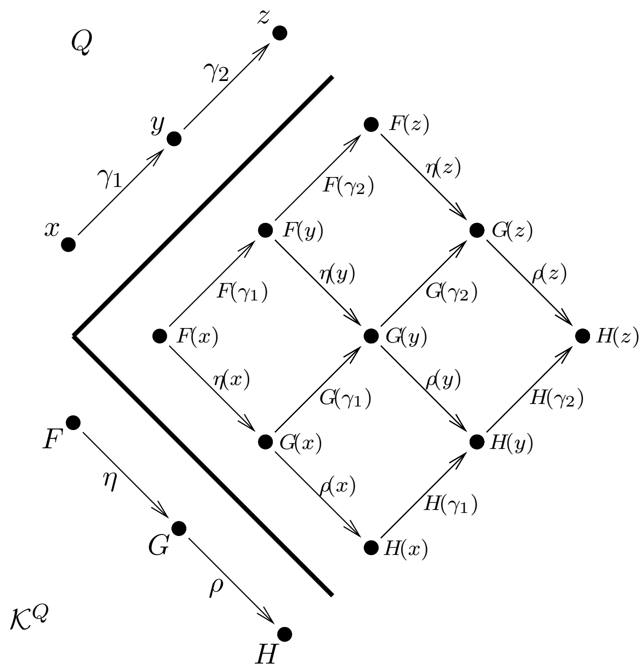
Every point $x \in \mathrm{Ob}(Q)$ gives rise to a functor $\tilde{x}$ defined by

$$\begin{aligned} \tilde{x} : \mathcal{K}^Q &\to \mathcal{K} \\ (F \xrightarrow{\eta} G) &\mapsto \left(F(x) \xrightarrow{\eta(x)} G(x) \right) \end{aligned}$$

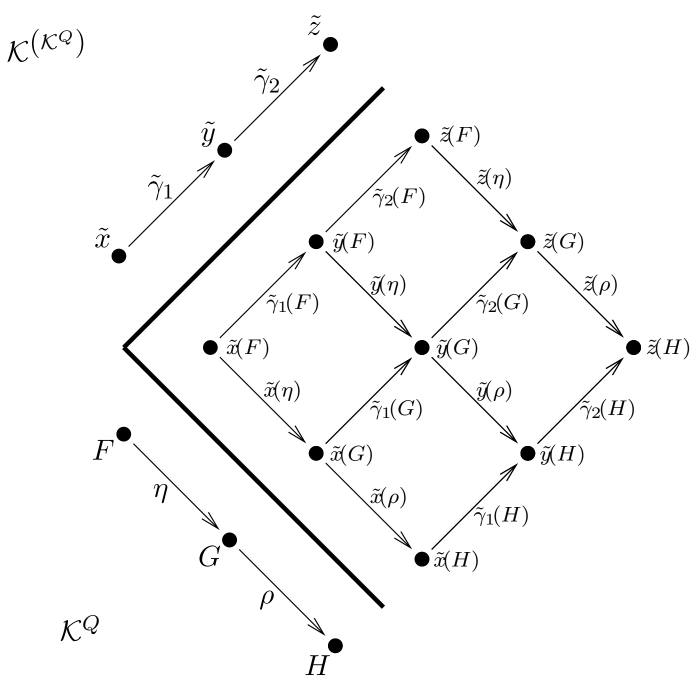
and every arrow $\gamma : x \to y$ in $\mathrm{Mor}(Q)$ gives rise to a natural transformation $\tilde{\gamma}$ between these functors

$$\begin{aligned} \tilde{\gamma} : \tilde{x} &\to \tilde{y} \\ \tilde{x}(F) &\xrightarrow{F(\gamma)} \tilde{y}(F) \ . \end{aligned}$$

This is best seen by looking at some naturality squares. Here is a 2-space Q together with a chain of three functors $F \to G \to H$ from Q to $\mathcal{K}$:



By using the definition of $\tilde{x}$ and $\tilde{\gamma}$ from above this can be relabelled equivalently to look like a 2-space with three points F, G and H and a chain $\tilde{x} \to \tilde{y} \to \tilde{z}$ of three functors from this to $\mathcal{K}$:



This duality is at the heart of the categorified Gelfand-Naimark theorem [97].

4.4.2 Vector 2-Bundles

Physical fields are in general not just functions on parameter space (spacetime/worldvolume) but are **sections of fiber bundles** over parameter space. Similarly, the wave function itself is in general not a function on configuration space, but a section of some bundle over configuration space. A central theme in categorified SQM is therefore necessarily that of 2-bundles.

It is well known (and reviewed in [98, 99]) that states of open strings ending on D-branes are described by certain **derived categories** (of coherent sheaves or of quiver representations). In §4.4.2.4 (p.89) it is discussed how this might naturally be interpretable in terms of categorified wave functions taking values in a **line 2-bundle**.

There are several equivalent descriptions of ordinary vector bundles. It turns out that the categorification depends on which description one starts with.

When using the definition which says that the typical fiber of a vector bundle is a vector space one ends up with categorifying the concept of a vector space itself. This was done in [40]. The **2-vector spaces** obtained this way are the right concept for instance for discussing Lie 2-algebras (§3.1.3 (p.42)) but they do not seem to give rise to an interesting notion of vector 2-bundle.

The definition of vector bundles most natural in the NCG context is that saying that a vector bundle $E \to M$ is a finitely generated projective module of the C^*-algebra $\mathbb{C}^M$ of complex-valued continuous functions on M. Categorifying this definition amounts to categorifying $\mathbb{C}^M$ such that the result is what should be called a 2-ring.

4.4.2.1 Vector 2-Bundles as 2-Modules. Aspects of this problem have notably been addressed in [97]. There it was argued that a good categorification of $\mathbb{C}^M$ is a functor category Hilb^Q, where Q is some base 2-space replacing M and Hilb is the category of Hilbert spaces, replacing $\mathbb{C}$.

A slightly different but very similar idea is used in [100], where instead of the category Hilb the category Vect is used. This amounts simply to forgetting about the scalar product.

The crucial point is that the tensor product in Vect (Hilb) makes Vect^Q (Hilb^Q) into a monoidal category and indeed, at least in the case studied in [97], into something that deserves to be called a **2-algebra**.

In the spirit of this concept of categorified function algebras the authors of [19] defined a vector 2-bundle to be something that locally is similar to a bundle whose typical fiber looks like Vect^n, for some integer n.

Equivalence classes of ordinary vector bundles are described by K-theory. Therefore one would expect that equivalence classes of vector 2-bundles are described by some categorification of K-theory, which perhaps should be related to the elliptic genus.

In [19] however it was found that equivalence classes of the vector 2-bundles as defined there are not quite described by elliptic cohomology, even though by something the authors call a *form* of elliptic cohomology.

On the other hand, this is maybe not too surprising. One should note that Vect^Q is more like a categorification of functions taking values in the natural numbers, than in the complex numbers (compare the discussion in [25]). In particular, there are no additive or

multiplicative inverses in Vect^Q. Due to that the "transition functions" in [19] are in general not invertible, for instance. This should mean that there must be a categorification of the notion "vector bundle" which more faithfully captures the crucial properties of ordinary vector bundles.

4.4.2.2 How to categorify function algebras? The above disucssion suggests that some more thoughts on the "right" categorification of function algebras is in order before vector 2-bundles can be addressed. One possibility to improve on Vect^Q might be the following:

We had observed that Vect^Q is lacking additive and multiplicative inverses. Hence we could try to enlarge Vect^Q by including such inverses, in a way similar to how one gets from the natural numbers to the integers and then the rational numbers.

In order to discuss this it turns out to be helpful to circumvent a couple of problems for the moment by restricting attention to base 2-spaces Q whose sets of points and arrows are *finite*. In particular, restrict attention to categories $Q = C_{(V,E)}$ which are free categories over finite directed graphs (V, E) (*cf.* §4.3.1.1 (p.73)). This serves as the categorification of the concept of a space consisting of a finite number of points.

In this case it is a simple fact that the functor category Vect^Q is the same as the category $KQ-\mathrm{Mod}$ of (left, say) modules of the **path algebra** KQ of Q,

$$\mathrm{Vect}^Q = KQ-\mathrm{Mod}\,.$$

Here the path algebra KQ is the algebra freely generated by the set of morphisms in Q with the product between these generators defined to be their composition when composable and zero otherwise.

This equivalent reformulation suggests to use the tensor product over KQ in order to form a monoidal category. By this reasoning we are led to include multiplication and multiplicative inverses by going from $KQ-\mathrm{Mod}$ to $KQ-\mathrm{Mod}-KQ$, the category of KQ *bimodules* (over K). The multiplicative inverses in $KQ-\mathrm{Mod}-KQ$ give rise to a group known as the *Picard group* of KQ.

(In a more general context one might of course want to consider different algebras A, B and the category $A-\mathrm{Mod}-B$. By left monoidal multiplication the weakly invertible elements $T \in A$–Mod–B give rise (if they exist) to a 'tilting equivalence' between A–Mod and Mod$-B$, in which case A and B are Morita equivalent.)

In a next step this category should be enlarged to allow a notion of subtraction. This again implies that given any object b there should be a way to 'decompose' it into abjects a and c. A diagramatic way to do this is by means of an exact sequence $a \to b \to c$. A slightly more general concept than this is that of a *distinguished triangle* in a triangulated category.

Using a triangulated category together with a **stability condition** on it [101, 102] subtraction is implemented by taking direct sums and then projecting the result onto the subset of objects which are **stable** with respect to this stability condition.

Now, a triangulated category is naturally obtained from $KQ-$Mod by passing to its **derived category** $\mathbf{D}(KQ-\text{Mod})$. The derived category $\mathbf{D}(C)$ of any additive category C is like the category $\mathbf{Ch}(C)$ of chain complexes in C but modulo some identifications.

Hence we should choose a stability condition on $\mathbf{D}(KQ-\text{Mod})$ and also pass from $KQ-\text{Mod}-KQ$ to its derived category. $\mathbf{D}(KQ-\text{Mod}-KQ)$.

(The weakly invertible objects in $\mathbf{D}(KQ-\text{Mod}-KQ)$ are known as *two-sided tilting complexes* and their isomorphism classes form a group known as the *derived Picard group* of KQ [103, 104, 105, 106].)

4.4.2.3 Vector 2-Bundles as $\mathbf{D}(A-\text{Mod}-A)$-modules. This way we have arrived at the proposal that a 'good' categorification of an ordinary function algebra on the set $\text{Ob}(Q)$ with values in K would be to replace $\text{Ob}(Q)$ by Q and the function algebra by the monoidal category $\mathbf{D}(KQ-\text{Mod}-KQ)$. Modules of this would locally look like $(\mathbf{D}(KQ-\text{Mod}))^n$ for some integer n. These would be our proposed vector 2-bundles over Q.

One might be worried that by going to derived categories the simple idea that a categorified function on $\text{Ob}(Q)$ is a functor on Q is lost. However, this is not the case. Namely, a chain complex of functors $Q \to \text{Vect}$ is the same as a functor $Q \to \mathbf{Ch}(\text{Vect})$,

$$\mathbf{Ch}\big(\text{Vect}^Q\big) = (\mathbf{Ch}(\text{Vect}))^Q$$

as the following diagram illustrates.

$$\begin{tikzcd}
z \\
y \arrow[u, "\gamma_2"] \\
x \arrow[u, "\gamma_1"]
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] & F^0(z) \arrow[r, "d_1(z)"] & F^1(z) \arrow[r, "d_2(z)"] & F^2(z) \arrow[r] & 0 \\
0 \arrow[u] \arrow[r] & F^0(y) \arrow[u, "F^0(\gamma_2)"'] \arrow[r, "d_1(y)"] & F^1(y) \arrow[u, "F^1(\gamma_2)"'] \arrow[r, "d_2(y)"] & F^2(y) \arrow[u, "F^2(\gamma_2)"'] \arrow[r] & 0 \arrow[u] \\
0 \arrow[u] \arrow[r] & F^0(x) \arrow[u, "F^0(\gamma_1)"'] \arrow[r, "d_1(x)"] & F^1(x) \arrow[u, "F^1(\gamma_1)"'] \arrow[r, "d_2(x)"] & F^2(x) \arrow[u, "F^2(\gamma_1)"'] \arrow[r] & 0 \arrow[u]
\end{tikzcd}$$

Here $x \xrightarrow{\gamma_1} y \xrightarrow{\gamma_2} z$ is a morphism in Q. The F on the right can be either regarded as giving a functor $Q \to \mathbf{Ch}\text{Vect}$ or a chain complex

$$0 \to F^1 \xrightarrow{d_1} F^1 \xrightarrow{d_2} F^2 \to 0$$

of functors $F^i : Q \to \text{Vect}$. That this diagram commutes can be regarded as a consequence of the definition of morphisms between functors (natural transformations) or as a consequence of the definition of chain maps. Moreover, due to the rule for 'vertical' composition of natural transformations (which is really going horizontal in this figure), we

have $d_1(x) \circ d_2(x) = 0$ and similarly for y and z irrespective whether we regard this as a chain complex of functors or as a single functor into $\mathbf{Ch}(\mathrm{Vect})$.

This means that a **vector 2-bundle** E according to the above proposal would be a 2-bundle with typical fiber $(\mathbf{Ch}\mathrm{Vect})^n$, i.e. one with local 2-trivializations

$$\begin{tikzcd}
p^{-1}U_i \arrow[rr, "t_i"] \arrow[dr, "p|_{p^{-1}U_i}"'] & & U_i \times (\mathbf{Ch}\mathrm{Vect})^n \arrow[dl] \\
& U_i &
\end{tikzcd}$$

(*cf.* §3.2 (p.46)).

A 2-section of this bundle resticted to $U_i \simeq Q_i$ would be a functor from Q_i to $(\mathbf{Ch}\mathrm{Vect})^n$, i.e. an element in

$$\mathrm{Ob}\Big(\big((\mathbf{Ch}\mathrm{Vect})^{Q_i}\big)^n\Big) = \mathrm{Ob}((\mathbf{D}(KQ_i{-}\mathrm{Mod}))^n)\ .$$

On double overlaps $U_{ij} = Q_i \cap Q_j$ the 2-transition

$$\bar{t}_i \circ t_j|_{U_{ij}} : Q_{ij} \times (\mathbf{Ch}\mathrm{Vect})^n \to Q_{ij} \times (\mathbf{Ch}\mathrm{Vect})^n$$

must be an invertible $n \times n$ matrix of two-sided tilting complexes in $\mathbf{D}(KQ{-}\mathrm{Mod}{-}KQ)$.

More precisely, such a matrix should act componentwise by the usual formula for matrix multiplication using the derived product operation of $\mathbf{D}(KQ_{ij}{-}\mathrm{Mod}{-}KQ_{ij})$ on $\mathbf{D}(KQ_{ij}{-}\mathrm{Mod})$ and the direct sum operation in $\mathbf{D}(KQ_{ij}{-}\mathrm{Mod})$ followed by a stability projection using the stability condition that we have chosen on $\mathbf{D}(KQ_{ij}{-}\mathrm{Mod})$ (in §4.4.2.2 (p.87)). These matrices would supersede the non-invertible transition matrices in [19].

The tractable special case of a **line 2-bundle** (i.e. a vector 2-bundle with $n = 1$) is already quite interesting:

4.4.2.4 Derived Category Description of D-Branes. If the general relation between categorification and stringification mentioned at the beginning of §I (p.7), as well as the notion of vector 2-bundle in §4.4.2.3 (p.88) are any good, then a 2-section of a line 2-bundle as described above should describe states of string, somehow.

And this indeed turns out to be the case.

By the above reasoning a 2-section of a line 2-bundle is locally an object of $\mathbf{D}(KQ{-}\mathrm{Mod}) = \mathbf{D}(\mathrm{Vect}^Q)$. This is indeed known to describe states of string stretched between D-branes, as reviewed in [98, 99].

In this context the path category Q, or rather its underlying graph, is really a quiver diagram encoding the precise nature of the moduli space of the effective field theory on these D-branes, or equivalently the nature of the transverse 'compact' dimensions. Hence it might seem that the analogy argued for above breaks down in that Q is not in any sense a categorified spacetime. But it turns out that viewed from a suitable perspective Q does have to be identified with a latticized spacetime after all, an effect known as **dimensional**

deconstruction. In particular, when Q is like $\mathbb{Z}_\infty$ or $\mathbb{Z}_\infty \times \mathbb{Z}_\infty$ it describes [107] two compactified dimensions of theories on 5-branes where 2-bundles are expected to play a role.

As discussed in §4.4.2.3 (p.88), gauge transformations of such a 2-section of a line 2-bundle have to be elements of the weak Picard 2-group inside $\mathbf{D}(KQ-\mathrm{Mod}-KQ)$. And indeed, these elements are known to describe duality transformations on these configurations, known as (fractional) Seiberg duality.

The relation of Seiberg duality to tilting equivalences is discussed in section 5.4 of [108]. Its relation to monodromies in moduli space, which goes back to [109] and others, is briefly reviewed in the introduction of [110].

5. Conclusion

The work we have presented consists of two parts. One is concerned with lifting supersymmetric quantum mechanics and its deformation theory to loop space and looking at string theory from this point of view. The other is concerned with constructing a global framework for this by categorifying the notion of globally defined parallel transport and holonomy for (world)lines to obtain globally defined 2-holonomy and parallel transport for (world)surfaces.

In the last subsection, §4.4 (p.83), we sketched some observations that are suggestive of a deeper principle behind this. Based on the fact that supersymmetric quantum mechanics is essentially nothing but spectral geometry of configuration space in terms of spectral triples, the above considerations suggest a grand scheme where stringy physics arises as a *categorified* supersymmetric quantum mechanics, or a categorified spectral geometry of stringy configuration space. While we did not attempt to seriously tackle this idea in its entirety, it serves as a motivating principle for the investigations and results that we presented.

We started by elaborating on known reformulations of the super Virasoro constraints of the RNS supersting as Dirac-Kähler operators on loop space, and showed how the known deformation principle for supersymmetric quantum mechanics allows to describe, when lifted to loop space, background fields for the superstring in terms of algebraic deformations. Without going into any mathematical rigour concerning this point we pointed out how this can be understood as a deformation of spectral triples for loop space geometry.

In the process of studying how D-branes, expressed in terms of boundary states, fit into this picture, we studied aspects of the so-called Pohlmeyer invariants, which are functionals on loop space that commute with all (super) Virasoro constraints. After (re)discovering the relation of these invariants to the more popular DDF invariants, we related them to the boundary states that describe D-branes with nonabelian gauge fields turned on.

This lead to the observation that a formally very natural generalization of these boundary states gives rise to a nonabelian connection 1-form on loop space with certain curious propertie, that were previously encountered in the context of categorified gauge theory.

Motivated by this we stepped back and began to investigate the global picture that this construction would be part of, showing that these connection 1-forms are precisely those that give rise to what we call 2-connections with 2-holonomy in strict 2-bundles. This was done in collaboration with John Baez.

In particular, we showed how the well-known cocycle relations for nonabelian gerbes with connection and curving are reproduced using this theory, thereby providing a useful alternative point of view on these structures, maybe in some sense a more transparent one.

This allowed us to construct what was previously known only in the case of abelian gerbes or else in the nonabelian but simplicial and globally trivial case, namely globally well-defined nonabelian 2-holonomy in possibly nontrivial 2-bundles.

We remarked that with such a 2-holonomy in hand there is an obvious way to generalize the usual action functional for strings coupled to an abelian 2-form (described by the abelian 2-holonomy of an abelian gerbe) to the nonabelian case. To the best of our knowledge this

gives for the first time a serious candidate formalism for capturing the dynamics of strings coupled to nonabelian 2-form fields. But here we did not attempt to go into any details concerning possible relations of such nonabelian action functionals to known or expected physics of such nonabelian 2-form fields.

On the other hand, in a collaboration with John Baez, Alissa Crans and Danny Stevenson, we could show that there are 2-bundles whose structure 2-group is related to the group $\mathrm{String}(n)$, which have apparently all the necessary properties to describe the global dynamics of spinning strings, i.e. which capture the global issues of spinors on loop space. There are several indications that our concept of 2-holonomy for these 2-bundles is closely related to the work on spinning strings by Stolz and Teichner, but here we only gave some hints concerning this point.

A largely complementary way to describe p-bundles with p-connection is their "differential" formulation. This gives rise to a nonabelian and weakened generalization of the well-known Deligne hypercohology description of abelian gerbes, which is a powerful formalism for deriving linearized cocycle conditions and gauge transformation laws in general semistrict p-bundles with p-connection. Moreover, this formalism is well adapted to the study of the dynamics of nonabelian p-form field theories, at least perturbatively. We end our present discussion with an analysis of the nonabelian Deligne hypercohomology description of the semistrict differential version of the above 2-bundles with structure 2-group related to $\mathrm{String}(n)$.

"One cannot help but
feel that there are many
beautiful secrets hidden
in loop space."
A. Polyakov [111]

Part II

SQM on Loop Space

In §6 the RNS superstring is approached from the point of view of supersymmetric quantum mechanics on loop space and, background fields are related to deformations. This is taken from [27]. For special cases of deformations this leads to the discussion of worldsheet invariants and boundary states in §7 (p.132), which is taken from [28, 35]. A generalization of such boundary states is shown in §8 (p.161) to give rise to the nonabelian local connections on loop space which then motivate the developments in part III.

6. Deformations and Background Fields

6.1 Introduction

Supersymmetric field theories look like Dirac-Kähler systems when formulated in Schrödinger representation. This has been well studied in the special limits where only a finite number of degrees of freedom are retained, such as the semi-classical quantization of solitons in field theory (see e.g. [112] for a brief introduction and further references). That this phenomenon is rooted in the general structure of supersymmetric field theory has been noted long ago in the second part of [3] (see also the second part of [4]). For 2 dimensional superconformal field theories describing superstring worldsheets a way to exploit this fact for the construction of covariant target space Hamiltonians (applicable to the computation of curvature corrections of string spectra in nontrivial backgrounds) has been proposed in [113]. In the construction of these Hamiltonians a pivotal role is played by a new method for obtaining functional representations of superconformal algebras (corresponding to nontrivial target space backgrounds) by means of certain deformations of the superconformal algebra.

In [113] the focus was on deformations which induce Kalb-Ramond backgrounds and only the 0-mode of the superconformal algebra was considered explicitly (which is sufficient for the construction of covariant target space Hamiltonians). Here this deformation technique is developed in more detail for the full superconformal algebra and for all massless bosonic string background fields. Other kinds of backgrounds can also be incorporated in principle and one goal of this discussion is to demonstrate the versatility of the new deformation technique for finding explicit functional realizations of the two-dimensional superconformal algebra.

The setting for our formalism is the representation of the superconformal algebra on the exterior bundle over loop space (the space of maps from the circle into target space) by means of K-deformed exterior (co)derivatives $\mathbf{d}_K$, ${\mathbf{d}^\dagger}_K$, where K is the Killing vector field on loop space which induces loop reparameterizations.

The key idea is that the form of the superconformal algebra is preserved under the deformation[2]

$$\begin{aligned} \mathbf{d}_K &\to e^{-\mathbf{W}}\, \mathbf{d}_K\, e^{\mathbf{W}} \\ {\mathbf{d}^\dagger}_K &\to e^{\mathbf{W}^\dagger}\, {\mathbf{d}^\dagger}_K\, e^{-\mathbf{W}^\dagger} \end{aligned} \tag{6.2}$$

if $\mathbf{W}$ is an even graded operator that satisfies a certain consistency condition.

[2]Throughout this section we use the term "deformation" to mean the operation (6.2) on the superconformal generators, the precise definition of which is given in §6.3.2 (p.107). These "deformations" are actually *isomorphisms* of the superconformal algebra, but affect its representations in terms of operators on the exterior bundle over loop space. In the literature one finds also other usages of the word "deformation" in the context of superalgebras, for instance for describing the map where the superbrackets $[\cdot,\cdot]_\iota$ are transformed as

$$[A,B]_\iota \to [A,B]_\iota + \sum_{t=1}^{\infty} \omega_i(A,B)\, t^i \tag{6.1}$$

with $\omega_i(A,B)$ elements of the superalgebra and t a real number (see [114]).

The canonical (functional) form of the superconformal generators for all massless NS and NS-NS backgrounds can neatly be expressed this way by deformation operators $\mathbf{W}$ that are bilinear in the fermions, as will be shown here. It turns out that there is one further bilinear in the fermions which induces a background that probably has to be interpreted as the RR 2-form as coupled to the D-string.

It is straightforward to find further deformation operators and hence further backgrounds. While the normal ordering effects which affect the superconformal algebras and which would give rise to equations of motion for the background fields are not investigated here, there is still a consistency condition to be satisfied which constrains the admissible deformation operators.

This approach for obtaining new superconformal algebras from existing ones by applying deformations is similar in spirit, but rather complementary, to the method of '*canonical deformations*' studied by Giannakis, Evans, Ovrut, Rama, Freericks, Halpern and others [115, 116, 117, 118, 119]. There, the superconformal generators T and G of one chirality are deformed to lowest order as

$$\begin{aligned} T(z) &\to T(z) + \delta T(z) \\ G(z) &\to G(z) + \delta G(z)\ . \end{aligned} \tag{6.3}$$

Requiring the deformed generators to satisfy the desired algebra to first order shows that δT and δG must be bosonic and fermionic components of a weight 1 worldsheet superfield. (An adaption of this procedure to deformations of the BRST charge itself is discussed in [120]. Another related discussion of deformations of BRST operators is given in [121].)

The advantage of this method over the one discussed in the following is that it operates at the level of quantum SCFTs and has powerful CFT tools at its disposal, such as normal ordering and operator product expansion. The disadvantage is that it only applies perturbatively to first order in the background fields, and that these background fields always appear with a certain gauge fixed.

On the other hand, the deformations discussed here which are induced by $\mathbf{d}_K \to e^{-\mathbf{W}}\mathbf{d}_K e^{\mathbf{W}} \sim e^{-\mathbf{W}}(iG + \bar{G})e^{\mathbf{W}}$ preserve the superconformal algebra for arbitrarily large perturbations $\mathbf{W}$. The drawback is that normal ordering is non-trivially affected, too, and without further work the resulting superconformal algebra is only available on the level of (bosonic and fermionic) Poisson brackets.

We show in §6.3.4.2 (p.117) that when restricted to first order the deformations that we are considering reproduce the theory of canonical deformations (6.3).

Our deformation method is also technically different from but related to the *marginal deformations* of conformal field theories (see [122] for a review and further references), where one sends the correlation function $\langle A\rangle$ of some operator A to the deformed correlation function

$$\langle A\rangle^{\lambda} \equiv \langle A \exp\left(\sum_i \lambda_i \int \mathcal{O}_i \,\mathrm{dvol}\right)\rangle\,, \tag{6.4}$$

where $\mathcal{O}_i$ are fields of conformal weight 1. This corresponds to adding the integral over a field of unit weight to the action. How this relates to the algebraic deformations of the superconformal algebra considered here is discussed in §6.3.4.1 (p.115).

The method discussed here generalizes the transformations studied in [8], where strings are regarded from the non-commutative geometry perspective. The main result of this approach (which goes back to [123] and [124]) is that T-duality as well as mirror symmetry can nicely be encoded by means of automorphisms of the vertex operator algebra. In terms of the above notation such automorphisms correspond to deformations induced by *anti-Hermitean* $\mathbf{W}^\dagger = -\mathbf{W}$, which induce pure gauge transformations on the algebra.

The analysis given here generalizes the approach of [8] in two ways: First, the use of Hermitean $\mathbf{W}$ in our formalism produces backgrounds which are not related by string dualities. Second, by calculating the functional form of the superconformal generators for these backgrounds we can study the action of anti-Hermitean $\mathbf{W}$ on these more general generators and find the transformation of the background fields under the associated target space duality.

In particular, we find a duality transformation which changes the sign of the dilaton and interchanges B- and C-form fields. It would seem that this must hence be related to S-duality. This question requires further analysis.

The structure of this section is as follows:

In §6.2 some technical preliminaries necessary for the following discussion are given. The functional loop space notation is introduced in §6.2.1, some basic facts about loop space geometry are discussed (§6.2.2), the exterior derivative and coderivative on that space are introduced (§6.2.2.2), and some remarks on isometries of loop space are given in §6.2.2.3.

This is then applied in §6.3 to the general analysis of deformations of the superconformal generators. First of all, the purely gravitational target space background is shown to be associated to the ordinary K-deformed loop space exterior derivative (§6.3.1). §6.3.2 then discusses how general continuous classical deformations of the superconformal algebra are obtained. As a first application, §6.3.3.1 shows how this can be used to get the previously discussed superconformal generators for purely gravitational backgrounds from those of flat space by a deformation.

Guided by the form of this deformation the following sections systematically list and analyze the deformations which are associated with the Kalb-Ramond, dilaton, and gauge field backgrounds (§6.3.3.2, §6.3.3.3, §6.3.3.4). It turns out (§6.3.3.5) that one further 2-form background can be obtained in a very similar fashion, which apparently has to be interpreted as the S-dual coupling of the D-string to the C_2 2-form background.

After having understood how the NS-NS backgrounds arise in our formalism we turn in §6.3.4 (p.115) to a comparison of the method presented here with the well-known 'canonical deformations', which are briefly reviewed in §6.3.4.1 (p.115). In §6.3.4.2 (p.117) it is shown how these canonical deformations are reproduced by means of the methods discussed here

and how our deformation operator $\mathbf{W}$ relates to the vertex operators of the respective background fields.

Next the inner relations between the various deformations found are further analyzed in §6.4. First of all §6.4.1 demonstrates how $\mathbf{d}_K$-exact deformation operators yield target space gauge transformations. Then, in §6.4.2 the well known realization of T-duality as an algebra isomorphism is adapted to the present context, and in §6.4.2.2 the action of a target space duality obtained from a certain modified algebra isomorphism on the various background fields is studied. It turns out that there are certain similarities to the action of loop space Hodge duality, which is discussed in §6.4.3.

The appendix lists some results from the canonical analysis of the D-string action, which are needed in the main text.

6.2 Loop Space

In this section the technical setup is briefly established. The 0-mode $\mathbf{d}_K$ of the sum of the left- and the rightmoving supercurrents is represented as the K-deformed exterior derivative on loop space. Weak nilpotency of this K-deformed operator (namely nilpotency up to reparameterizations) is the essential property which implies that the modes of $\mathbf{d}_K$ and its adjoint generate a superconformal algebra. In this sense the loop space perspective on superstrings highlights a special aspect of the super Virasoro constraint algebra which turns out to be pivotal for the construction of classical deformations of that algebra.

The kinematical configuration space of the closed *bosonic* string is loop space $\mathcal{L}M$, the space of parameterized loops in target space M. As discussed in §2.1 of [113] the kinematical configuration space of the closed *superstring* is therefore the superspace over $\mathcal{L}M$, which can be identified with the 1-form bundle $\Omega^1(\mathcal{L}M)$. Superstring states in Schrödinger representation are super-functionals on $\Omega^1(\mathcal{L}M)$ and hence section of the form bundle $\Omega(\mathcal{L}M)$ over loop space.

The main technical consequence of the infinite dimensionality are the well known divergencies of certain objects, such as the Ricci-Tensor and the Laplace-Beltrami operator, which inhibit the naive implementation of quantum mechanics on $\mathcal{L}M$. But of course these are just the well known infinities that arise, when working in the Heisenberg (CFT) instead of in the Schrödinger picture, from operator ordering effects, and which should be removed by imposing normal ordering. Since the choice of Schrödinger or Heisenberg picture is just one of language, the same normal ordering (now expressed in terms of functional operators instead of Fock space operators) takes care of infinities in loop space. We will therefore not have much more to say about this issue here. The main result of this section are various (deformed) representations of the super-Virasoro algebra on loop space (corresponding to different spacetime backgrounds), and will be derived in their classical (Poisson-bracket) form without considering normal ordering effects.

A mathematical discussion of aspects of loop space can for instance be found in [125, 126]. A rigorous treatment of some of the objects discussed below is also given in [127].

6.2.1 Definitions

Let (M, g) be a pseudo-Riemannian manifold, the target space, with metric g, and let $\mathcal{L}M$ be its loop space consisting of smooth maps of the *parameterized* circle with parameter $\sigma \sim \sigma + 2\pi$ into M:

$$\mathcal{L}M \equiv C^\infty\big(S^1, M\big)\ . \tag{6.5}$$

The tangent space $T_X\mathcal{L}M$ of $\mathcal{L}M$ at a loop $X : S^1 \to M$ is the space of vector fields along that loop. The metric on M induces a metric on $T_X\mathcal{L}M$: Let $g(p) = g_{\mu\nu}(p)\, dx^\mu \otimes dx^\nu$ be

the metric tensor on M. Then we choose for the metric on $\mathcal{L}M$ at a point X the mapping

$$\begin{aligned} T_X\mathcal{L}M \times T_X\mathcal{L}M &\to \mathbb{R} \\ (U,V) \mapsto U\cdot V &= \int_0^{2\pi} d\sigma\, g(X(\sigma))(U(\sigma)\,, V(\sigma)) \\ &= \int_0^{2\pi} d\sigma\, g_{\mu\nu}(X(\sigma))\, U^\mu(\sigma)\, V^\nu(\sigma)\ . \end{aligned} \tag{6.6}$$

For the intended applications $T\mathcal{L}M$ is actually too small, since there will be need to deal with distributional vector fields on loop space. Therefore one really considers $\bar{T}\mathcal{L}M$, the completion of $T\mathcal{L}M$ at each point X with respect to the norm induced by the inner product (6.6).) For brevity, whenever we refer to "loop space" in the following, we mean $\mathcal{L}M$ equipped with the metric (6.6). Also, the explicit integration region $\sigma \in (0, 2\pi)$ will be implicit in the following.

To abbreviate the notation, let us introduce formal multi-indices (μ, σ) and write equivalently

$$U^\mu(\sigma) \equiv U^{(\mu,\sigma)} \tag{6.7}$$

for a vector $U \in T_X\mathcal{L}M$, and similarly for higher-rank tensors on loop space.

Extending the usual index notation to the infinite-dimensional setting in the obvious way, we also write:

$$\int U^\mu(\sigma)\, V_\mu(\sigma) \equiv U^{(\mu,\sigma)} V_{(\mu,\sigma)}\ . \tag{6.8}$$

For this to make sense we need to know how to "shift" the continuous index σ. Because of

$$\int d\sigma\, g_{\mu\nu}(X(\sigma))\, U^\mu(\sigma)\, V^\nu(\sigma) = \int d\sigma\, d\sigma'\, \delta\big(\sigma,\sigma'\big)\, g_{\mu\nu}(X(\sigma))\, U^\mu(\sigma)\, V^\nu\big(\sigma'\big)$$

it makes sense to write the metric tensor on loop space as

$$G_{(\mu,\sigma)(\nu,\sigma')}(X) \equiv g_{\mu\nu}(X(\sigma))\, \delta\big(\sigma,\sigma'\big)\ . \tag{6.9}$$

Therefore

$$\langle U, V\rangle = U^{(\mu,\sigma)} G_{(\mu,\sigma)(\nu,\sigma')} V^{(\nu,\sigma')} \tag{6.10}$$

and

$$\begin{aligned} V_{(\mu,\sigma)} &= G_{(\mu,\sigma)(\nu,\sigma')} V^{(\nu,\sigma')} \\ &= V_\mu(\sigma)\ . \end{aligned} \tag{6.11}$$

Consequently, it is natural to write

$$\delta\big(\sigma,\sigma'\big) \equiv \delta_\sigma^{\sigma'} = \delta_{\sigma'}^{\sigma} = \delta_{\sigma,\sigma'} = \delta^{\sigma,\sigma'}\ . \tag{6.12}$$

A (holonomic) basis for $T_X\mathcal{L}M$ may now be written as

$$\partial_{(\mu,\sigma)} \equiv \frac{\delta}{\delta X^\mu(\sigma)}\,, \tag{6.13}$$

where the expression on the right denotes the functional derivative, so that

$$\begin{aligned} \partial_{(\mu,\sigma)} X^{(\nu,\sigma')} &= \delta_{(\mu,\sigma)}^{(\nu,\sigma')} \\ &= \delta_\mu^\nu\, \delta\big(\sigma,\sigma'\big)\ . \end{aligned} \tag{6.14}$$

By analogy, many concepts known from finite dimensional geometry carry over to the infinite dimensional case of loop spaces. Problems arise when traces over the continuous “index” σ are taken, like for contractions of the Riemann tensor, which leads to undefined diverging expressions. It is expected that these are taken care of by the usual normal-ordering of quantum field theory.

6.2.2 Differential Geometry on Loop Space

With the metric (6.9) on loop space in hand

$$G_{(\mu,\sigma)(\nu,\sigma')}(X) = g_{\mu\nu}(X(\sigma))\,\delta_{\sigma,\sigma'} \tag{6.15}$$

the usual objects of differential geometry can be derived for loop space. Simple calculations yield the Levi-Civita connection as well as the Riemann curvature, which will be frequently needed later on. The exterior algebra over loop space is introduced and the exterior derivative and its adjoint, which play the central role in the construction of the super-Virasoro algebra in §6.3.1 (p.104), are constructed in terms of operators on the exterior bundle. Furthermore isometries on loop space are considered, both the one coming from reparameterization of loops as well as those induced from target space. The former leads to the reparameterization constraint on strings, while the latter is crucial for the Hamiltonian evolution on loop space [113].

6.2.2.1 Basic geometric data. The inverse metric is obviously

$$G^{(\mu,\sigma)(\nu,\sigma')}(X) = g^{\mu\nu}(X(\sigma))\,\delta\big(\sigma,\sigma'\big)\ . \tag{6.16}$$

A vielbein field $\mathbf{e}^a = e^a{}_\mu \mathbf{d}x^\mu$ on M gives rise to a vielbein field $\mathbf{E}^{(a,\sigma)}$ on loop space:

$$E^{(a,\sigma)}{}_{(\mu,\sigma')}(X) \equiv e^a{}_\mu(X(\sigma))\,\delta^\sigma{}_{\sigma'} \tag{6.17}$$

which satisfies

$$\begin{aligned} E^{(a,\sigma)}{}_{(\mu,\sigma'')}E^{(b,\sigma)(\mu,\sigma'')} &= \eta^{ab}\delta^{\sigma,\sigma'} \\ &\equiv \eta^{(a,\sigma)(b,\sigma')} \end{aligned} \tag{6.18}$$

For the Levi-Civita connection one finds:

$$
\begin{aligned}
&\Gamma_{(\mu\sigma)(\alpha\sigma')(\beta\sigma'')}(X) \\
&= \frac{1}{2}\left(\frac{\delta}{\delta X^\mu(\sigma)} G_{(\alpha,\sigma')(\beta,\sigma'')}(X) + \frac{\delta}{\delta X^\beta(\sigma'')} G_{(\mu,\sigma)(\alpha,\sigma')}(X) - \frac{\delta}{\delta X^\alpha(\sigma')} G_{(\beta,\sigma'')(\mu,\sigma)}(X)\right) \\
&= \frac{1}{2}\left((\partial_\mu G_{\alpha\beta})\big(X(\sigma')\big)\,\delta\big(\sigma,\sigma'\big)\,\delta\big(\sigma',\sigma''\big) + (\partial_\beta G_{\mu\alpha})(X(\sigma))\,\delta\big(\sigma'',\sigma\big)\,\delta\big(\sigma,\sigma'\big)\right) \\
&\quad -\frac{1}{2}(\partial_\alpha G_{\beta\mu})\big(X(\sigma'')\big)\,\delta\big(\sigma',\sigma''\big)\,\delta\big(\sigma',\sigma\big) \\
&= \Gamma_{\mu\alpha\beta}(X(\sigma))\;\delta\big(\sigma,\sigma'\big)\,\delta\big(\sigma',\sigma''\big)\;,
\end{aligned}
\tag{6.19}
$$

and hence

$$
\Gamma_{(\mu,\sigma)}{}^{(\alpha,\sigma')}{}_{(\beta,\sigma'')}(X) \;=\; \Gamma_\mu{}^\alpha{}_\beta(X(\sigma))\;\delta\big(\sigma,\sigma'\big)\,\delta\big(\sigma',\sigma''\big)\;. \tag{6.20}
$$

The respective connection in an orthonormal basis is

$$
\begin{aligned}
\omega_{(\mu,\sigma)}{}^{(a\sigma')}{}_{(b,\sigma'')}(X) \;&=\; E^{(a,\sigma')}{}_{(\alpha,\rho)}(X)\left(\delta^{(\alpha,\rho)}_{(\beta,\rho')}\partial_{(\mu,\sigma)} + \Gamma_{(\mu,\sigma)}{}^{(\alpha,\rho)}{}_{(\beta,\rho')}(X)\right) E^{(\beta,\rho')}{}_{(b,\sigma')}(X) \\
&=\; \omega_\mu{}^a{}_b(X(\sigma))\,\delta\big(\sigma,\sigma'\big)\,\delta\big(\sigma',\sigma''\big)\;.
\end{aligned}
\tag{6.21}
$$

From (6.20) the Riemann tensor on loop space is obtained as

$$
\begin{aligned}
&R_{(\mu,\sigma_1)(\nu,\sigma_2)}{}^{(\alpha,\sigma_3)}{}_{(\beta,\sigma_4)}(X) \\
&= 2\frac{\delta}{\delta X^{[(\mu,\sigma_1)}}\Gamma_{(\nu,\sigma_2)]}{}^{(\alpha,\sigma_3)}{}_{(\beta,\sigma_4)} + 2\Gamma_{[(\mu,\sigma_1)}{}^{(\alpha,\sigma_3)}{}_{|(X,\sigma_5)|}\Gamma_{(\nu,\sigma_2)]}{}^{(X,\sigma_5)}{}_{(\beta,\sigma_4)} \\
&= R_{\mu\nu}{}^\alpha{}_\beta(X(\sigma_1))\;\delta(\sigma_1,\sigma_2)\,\delta(\sigma_2,\sigma_3)\,\delta(\sigma_3,\sigma_4)\;.
\end{aligned}
\tag{6.22}
$$

The Ricci tensor is formally

$$
\begin{aligned}
R_{(\mu,\sigma)(\nu,\sigma')}(X) \;&=\; R_{(\kappa,\sigma'')(\mu,\sigma)}{}^{(\kappa,\sigma'')}{}_{(\nu,\sigma')}(X) \\
&=\; R_{\mu\nu}(X(\sigma))\,\delta\big(\sigma,\sigma'\big)\;\delta\big(\sigma'',\sigma''\big)\;,
\end{aligned}
\tag{6.23}
$$

which needs to be regularized. Similarly the curvature scalar is formally

$$
\begin{aligned}
R(X) \;&=\; R_{(\mu,\sigma)}{}^{(\mu,\sigma)}(X) \\
&=\; R(X(\sigma))\,\delta^\sigma_\sigma\delta\big(\sigma'',\sigma''\big)\;.
\end{aligned}
\tag{6.24}
$$

6.2.2.2 Exterior and Clifford algebra over loop space. The anticommuting fields $\mathcal{E}^{\dagger(\mu,\sigma)}$, $\mathcal{E}_{(\mu,\sigma)}$, satisfying the CAR

$$
\begin{aligned}
\left\{\mathcal{E}^{\dagger(\mu,\sigma)},\mathcal{E}^{\dagger(\nu,\sigma')}\right\} &= 0 \\
\left\{\mathcal{E}_{(\mu,\sigma)},\mathcal{E}_{(\nu,\sigma')}\right\} &= 0 \\
\left\{\mathcal{E}_{(\mu,\sigma)},\mathcal{E}^{\dagger(\nu,\sigma')}\right\} &= \delta^{(\mu,\sigma)}_{(\nu,\sigma')}\,,
\end{aligned}
\tag{6.25}
$$

are assumed to exist over loop space, in analogy with the creators and annihilators $\hat{c}^{\dagger\mu}$, $\hat{c}_\mu$ on the exterior bundle in finite dimensions as described in appendix A of [113]. (For

a mathematically rigorous treatment of the continuous CAR compare [126] and references given there.) From them the Clifford fields

$$\Gamma_\pm^{(\mu,\sigma)} \equiv \mathcal{E}^{\dagger(\mu,\sigma)} \pm \mathcal{E}^{(\mu,\sigma)} \tag{6.26}$$

are obtained, which satisfy

$$\begin{aligned}\left\{\Gamma_\pm^{(\mu,\sigma)}, \Gamma_\pm^{(\nu,\sigma')}\right\} &= \pm 2G^{(\mu,\sigma)(\nu,\sigma')} \\ \left\{\Gamma_\pm^{(\mu,\sigma)}, \Gamma_\mp^{(\nu,\sigma')}\right\} &= 0\,.\end{aligned} \tag{6.27}$$

Since the $\Gamma_\pm$ will be related to spinor fields on the string's worldsheet, we alternatively use spinor indices $A, B, \ldots \in \{1,2\} \simeq \{+,-\}$ and write

$$\left\{\Gamma_A^{(\mu,\sigma)}, \Gamma_B^{(\nu,\sigma')}\right\} = 2s_A\delta_{AB}G^{(\mu,\sigma)(\nu,\sigma')}\,. \tag{6.28}$$

Here s_A is defined by

$$s_+ = +1\,, \quad s_- = -1\,. \tag{6.29}$$

The above operators will frequently be needed with respect to some orthonormal frame $E^{(a,\sigma)}$:

$$\Gamma_A^{(a,\sigma)} \equiv E^{(a,\sigma)}{}_{(\mu,\sigma')}\Gamma_A^{(\mu,\sigma')}\,. \tag{6.30}$$

Just like in the finite dimensional case, the following derivative operators can now be defined:

The covariant derivative operator (*cf.* A.2 in [113]) on the exterior bundle over loop space is

$$\begin{aligned}\hat\nabla_{(\mu,\sigma)} &= \partial^c_{(\mu,\sigma)} - \Gamma_{(\mu,\sigma)}{}^{(\alpha,\sigma')}{}_{(\beta,\sigma'')}\mathcal{E}^{\dagger(\beta,\sigma'')}\mathcal{E}_{(\alpha,\sigma')} \\ &= \partial^c_{(\mu,\sigma)} - \int d\sigma'\, d\sigma''\Gamma_\mu{}^\alpha{}_\beta(X(\sigma))\,\delta(\sigma,\sigma')\,\delta(\sigma',\sigma'')\,\mathcal{E}^{\dagger\beta}(\sigma'')\,\mathcal{E}_\alpha(\sigma') \\ &= \partial^c_{(\mu,\sigma)} - \Gamma_\mu{}^\alpha{}_\beta(X(\sigma))\,\mathcal{E}^{\dagger\beta}(\sigma)\,\mathcal{E}_\alpha(\sigma)\end{aligned} \tag{6.31}$$

or alternatively

$$\hat\nabla_{(\mu,\sigma)} = \partial_{(\mu,\sigma)} - \omega_\mu{}^a{}_b(X(\sigma))\,\mathcal{E}^{\dagger b}(\sigma)\,\mathcal{E}_a(\sigma)\,. \tag{6.32}$$

One should note well the difference between the functional derivative $\partial^c_{(\mu,\sigma)}$ which commutes with the coordinate frame forms ($[\partial^c_{(\mu,\sigma)}\mathcal{E}^{\dagger\nu}] = 0$) and the functional derivative $\partial_{(\mu,\sigma)}$ which instead commutes with the ONB frame forms ($[\partial_{(\mu,\sigma)}\mathcal{E}^{\dagger a}] = 0$). See (A.29) of [113] for more details.

In terms of these operators the exterior derivative and coderivative on loop space read, respectively (A.39)

$$\begin{aligned}\mathbf{d} &= \mathcal{E}^{\dagger(\mu,\sigma)}\partial^c_{(\mu,\sigma)} \\ &= \mathcal{E}^{\dagger(\mu,\sigma)}\hat\nabla_{(\mu,\sigma)} \\ \mathbf{d}^\dagger &= -\mathcal{E}^{(\mu,\sigma)}\hat\nabla_{(\mu,\sigma)}\,.\end{aligned} \tag{6.33}$$

We will furthermore need the form number operator

$$\mathcal{N} = \mathcal{E}^{\dagger(\mu,\sigma)}\mathcal{E}_{(\mu,\sigma)} \tag{6.34}$$

as well as its *modes*: Let $\xi : S^1 \to \mathbb{C}$ be a smooth function then

$$\mathcal{N}_\xi \equiv \int d\sigma\, \xi(\sigma)\, \mathcal{E}^{\dagger\mu}(\sigma)\, \mathcal{E}_\mu(\sigma) \tag{6.35}$$

is the ξ-mode of the form number operator. Commuting it with the exterior derivative yields the modes of that operator:

$$\begin{aligned}
\mathbf{d}_\xi &\equiv [\mathcal{N}_\xi, \mathbf{d}] \\
&= \int d\sigma\, \xi(\sigma)\;\; \mathcal{E}^{\dagger\mu}(\sigma)\, \hat{\nabla}_\mu(\sigma) \\
\mathbf{d}^\dagger{}_\xi &\equiv -\left[\mathcal{N}_\xi, \mathbf{d}^\dagger\right] \\
&= -\int d\sigma\, \xi(\sigma)\;\; \mathcal{E}^{\mu}(\sigma)\, \hat{\nabla}_\mu(\sigma)\ .
\end{aligned} \tag{6.36}$$

These modes will play a crucial role in §6.3 (p.104).

6.2.2.3 Isometries. Regardless of the symmetries of the metric g on M, loop space $(\mathcal{L}M, G)$ has an isometry generated by the reparameterization flow vector field K, which is defined by:[3]

$$K^{(\mu,\sigma)}(X) = T\, X'^{\mu}(\sigma)\ . \tag{6.37}$$

(Here T is just a constant which we include for later convenience.) The flow generated by this vector field obviously rotates the loops around. Since the metric (6.15) is "diagonal" in the parameter σ, this leaves the geometry of loop space invariant, and the vector field K satisfies Killing's equation

$$G_{(\nu,\sigma')(X,\sigma'')}\nabla_{(\mu,\sigma)}K^{(X,\sigma'')} + G_{(\mu,\sigma)(X,\sigma'')}\nabla_{(\nu,\sigma')}K^{(X,\sigma'')} = 0\,, \tag{6.38}$$

as is readily checked.

The Lie-derivative along K is (see section A.4 of [113])

$$\begin{aligned}
\mathcal{L}_K &= \left\{\mathcal{E}^{\dagger(\mu,\sigma)}\partial^c_{(\mu,\sigma)}, \mathcal{E}_{(\nu,\sigma')}X'^{(\nu,\sigma')}\right\} \\
&= X'^{(\mu,\sigma)}\partial^c_{(\mu,\sigma)} + \mathcal{E}^{\dagger(\mu,\sigma)}\mathcal{E}_{(\nu,\sigma')}\delta'_{\sigma',\sigma} \\
&= X'^{(\mu,\sigma)}\partial^c_{(\mu,\sigma)} + \mathcal{E}^{\dagger\prime(\mu,\sigma)}\mathcal{E}_{(\mu,\sigma)}\ .
\end{aligned} \tag{6.39}$$

This operator will be seen to be an essential ingredient of the super-Virasoro algebra in §6.3 (p.104).

[3] Here and in the following a prime indicates the derivative with respect to the loop parameter σ: $X'(\sigma) = \partial_\sigma X(\sigma)$.

Apart from the generic isometry (6.37), every symmetry of the target space manifold M gives rise to a family of symmetries on $\mathcal{L}M$: Let v be any Killing vector on target space,

$$\nabla_{(\mu} v_{\nu)} = 0\,, \tag{6.40}$$

then every vector V on loop space of the form

$$V_\xi(X) = V_\xi^{(\mu,\sigma)}(X)\,\partial_{(\mu,\sigma)} \;\equiv\; v^\mu(X(\sigma))\,\xi^\sigma \partial_{(\mu,\sigma)}\,, \tag{6.41}$$

where $\xi^\sigma = \xi(\sigma)$ is some differentiable function $S^1 \to \mathbb{C}$, is a Killing vector on loop space. For the commutators one finds

$$\begin{aligned} [V_{\xi_1}, V_{\xi_2}] &= 0 \\ [V_\xi, K] &= V_{\xi'}\,. \end{aligned} \tag{6.42}$$

The reparameterization Killing vector K will be used to deform the exterior derivative on loop space as discussed in §2.1.1 of [113], and a target space induced Killing vector V_ξ will serve as a generator of parameter evolution on loop space along the lines of §2.2 of [113]. There it was found in equation (88) that the condition

$$[K, V_\xi] = 0 \tag{6.43}$$

needs to be satisfied for this to work. Due to (6.42) this means that one needs to choose $\xi = \text{const}$, i.e. use the integral lines of $V_{\xi=1}$ as the "time"-parameter on loop space. This is only natural: It means that every point on the loop is evolved equally along the Killing vector field v on target space.

6.3 Superconformal Generators for Various Backgrounds

We now use the loop space technology to show that the loop space exterior derivative deformed by the reparameterization Killing vector K gives rise to the superconformal algebra which describes string propagation in purely gravitational backgrounds. General deformations of this algebra are introduced and applying these we find representations of the superconformal algebra that correspond to all the massless NS and NS-NS background fields.

(Parts of this construction were already indicated in [113], but there only the 0-modes of the generators and only a subset of massless bosonic background fields was considered, without spelling out the full nature of the necessary constructions on loop space.)

6.3.1 Purely Gravitational Background

In this subsection it is described how to obtain a representation of the classical super-Virasoro algebra on loop space. For a trivial background the construction itself is relatively trivial and, possibly in different notation, well known. The point that shall be emphasized here is that the identification of super-Virasoro generators with modes of the deformed exterior(co-)derivative on loop space allows a convenient treatment of curved backgrounds as well as more general non-trivial background fields.

As was discussed in [113], §2.1.1 (which is based on [3, 4]), one may obtain from the exterior derivative and its adjoint on a manifold the generators of a global $D = 2$, $N = 1$ superalgebra by deforming with a Killing vector. The generic Killing vector field on loop space is the reparameterization generator (6.37). Using this to deform the exterior derivative and its adjoint as in equation (19) of [113] yields the operators

$$\begin{aligned} \mathbf{d}_K &\equiv \mathbf{d} + i\mathcal{E}_{(\mu,\sigma)} X'^{(\mu,\sigma)} \\ \mathbf{d}^\dagger{}_K &\equiv \mathbf{d}^\dagger - i\mathcal{E}^\dagger_{(\mu,\sigma)} X'^{(\mu,\sigma)} , \end{aligned} \tag{6.44}$$

(where for convenience we set $T = 1$ for the moment) which generate a *global* superalgebra. Before having a closer look at this algebra let us first enlarge it to a local superalgebra by considering the *modes* defined by

$$\begin{aligned} \mathbf{d}_{K,\xi} &\equiv [\mathcal{N}_\xi, \mathbf{d}^*_K] \\ \mathbf{d}^\dagger{}_{K,\xi^*} &\equiv -\left[\mathcal{N}_\xi, \mathbf{d}^\dagger{}^*_K\right] , \end{aligned} \tag{6.45}$$

where $\cdot^*$ is the complex adjoint and $\mathcal{N}_\xi$ is the ξ-mode of the form number operator discussed in (6.35). They explicitly read

$$\begin{aligned} \mathbf{d}_{K,\xi} &= \int d\sigma\; \xi(\sigma) \left(\mathcal{E}^{\dagger\mu}(\sigma)\, \partial^c_\mu(\sigma) + i\mathcal{E}_\mu(\sigma)\, X'^\mu(\sigma)\right) \\ \mathbf{d}^\dagger{}_{K,\xi} &= -\int d\sigma\; \xi(\sigma) \left(\mathcal{E}^{\mu}(\sigma)\, \nabla_\mu(\sigma) + i\mathcal{E}^\dagger_\mu(\sigma)\, X'^\mu(\sigma)\right) . \end{aligned} \tag{6.46}$$

Making use of the fact that $\mathbf{d}_{K,\xi}$ is actually independent of the background metric, it is easy to establish the algebra of these operators. We do this for the "classical" fields, ignoring normal ordering effects and the anomaly:

The anticommutator of the operators (6.45) with themselves defines the ξ-mode $\mathcal{L}_{K,\xi}$ of the Lie-derivative $\mathcal{L}_K$ along K:

$$\{\mathbf{d}_{K,\xi_1}, \mathbf{d}_{K,\xi_2}\} = 2i\mathcal{L}_{K,\xi_1\xi_2} , \tag{6.47}$$

where

$$\mathcal{L}_\xi = \int d\sigma\; \left(\xi(\sigma)\, X'^\mu(\sigma)\, \partial^c_\mu(\sigma) + \sqrt{\xi}\left(\sqrt{\xi}\mathcal{E}^{\dagger\mu}\right)'(\sigma)\, \mathcal{E}_\mu(\sigma)\right) . \tag{6.48}$$

We say that a field $A(\sigma)$ has *reparameterization weight* w if

$$\begin{aligned} \left[\mathcal{L}_\xi, A(\sigma)\right]_\iota &= \left(\xi A' + w\xi' A\right)(\sigma) \\ \left[\mathcal{L}_{\xi_1}, A_{\xi_2}\right]_\iota &= A_{(w-1)\xi_1'\xi_2 - \xi_1\xi_2'} , \end{aligned} \tag{6.49}$$

where $A_\xi \equiv \int d\sigma\; \xi A$. For the basic fields we find

$$\begin{aligned} w(X^\mu) &= 0 \\ w\left(X'^\mu\right) &= 1 \\ w\left(\partial^c_\mu\right) &= 1 \\ w\left(\Gamma^\mu_\pm\right) &= 1/2 . \end{aligned} \tag{6.50}$$

Because of $w(AB) = w(A) + w(B)$ it follows that $\mathbf{d}_{K,\xi}$ and $\mathbf{d}^\dagger{}_{K,\xi}$ are modes of integrals over densities of reparameterization weight $w = 3/2$. This implies in particular that

$$\begin{aligned} [\mathcal{L}_{\xi_1}, \mathbf{d}_{K,\xi_2}] &= \mathbf{d}_{K,(\frac{1}{2}\xi_1'\xi_2 - \xi_1\xi_2')} && (6.51)\\ [\mathcal{L}_{K,\xi_1}, \mathcal{L}_{K,\xi_2}] &= \mathcal{L}_{K,(\xi_1'\xi_2 - \xi_1\xi_2')}\,. && (6.52)\end{aligned}$$

By taking the adjoint of (6.47) and (6.51) (or by doing the calculation explicitly), analogous relations are found for $\mathbf{d}^\dagger{}_{K,\xi}$:

$$\begin{aligned} \left\{\mathbf{d}^\dagger{}_{K,\xi_1}, \mathbf{d}^\dagger{}_{K,\xi_2}\right\} &= 2i\mathcal{L}_{K,\xi_1\xi_2}\\ \left[\mathcal{L}_{K,\xi_1}, \mathbf{d}^\dagger{}_{K,\xi_2}\right] &= \mathbf{d}^\dagger{}_{K,(\frac{1}{2}\xi_1'\xi_2 - \xi_1\xi_2')}\,. \end{aligned} \qquad (6.53)$$

Equations (6.47), (6.51), and (6.53) give us part of the sought-after algebra. A very simple and apparently unproblematic but rather crucial step for finding the rest is to now define the *modes of the deformed Laplace-Beltrami operator* as the right hand side of

$$\left\{\mathbf{d}_{K,\xi_1}, \mathbf{d}^\dagger{}_{K,\xi_2}\right\} = \mathbf{\Delta}_{K,\xi_1\xi_2}\,. \qquad (6.54)$$

For this definition to make sense one needs to check that

$$\left\{\mathbf{d}_{K,\xi_1\xi_3}, \mathbf{d}^\dagger{}_{K,\xi_2}\right\} = \left\{\mathbf{d}_{K,\xi_1}, \mathbf{d}^\dagger{}_{K,\xi_2\xi_3}\right\}\,. \qquad (6.55)$$

It is easy to verify that this is indeed true for the operators as given in (6.46). However, in §6.3.2 (p.107) it is found that this condition is a rather strong constraint on the admissible perturbations of these operators, and the innocent looking equation (6.55) plays a pivotal role in the algebraic construction of superconformal field theories in the present context.

With $\mathbf{\Delta}_{K,\xi}$ consistently defined as in (6.54) all remaining brackets follow by using the Jacobi-identity:

$$\begin{aligned} \left[\frac{1}{2}\mathbf{\Delta}_{K,\xi_1}, \mathbf{d}_{K,\xi_2}\right] &= i\mathbf{d}^\dagger{}_{K,(\frac{1}{2}\xi_1'\xi_2 - \xi_1\xi_2')}\\ \left[\frac{1}{2}\mathbf{\Delta}_{K,\xi_1}, \mathbf{d}^\dagger{}_{K,\xi_2}\right] &= i\mathbf{d}_{K,(\frac{1}{2}\xi_1'\xi_2 - \xi_1\xi_2')}\\ \left[\frac{1}{2}\mathbf{\Delta}_{K,\xi_1}, \frac{1}{2}\mathbf{\Delta}_{K,\xi_2}\right] &= -\mathcal{L}_{K,(\xi_1'\xi_2 - \xi_1\xi_2')}\,. \end{aligned} \qquad (6.56)$$

In order to make the equivalence to the super-Virasoro algebra of the algebra thus obtained more manifest consider the modes of the K-deformed Dirac-Kähler operators on loop space:

$$\begin{aligned} \mathbf{D}_{K,\pm} &\equiv \mathbf{d}_K \pm \mathbf{d}^\dagger{}_K\\ &= \Gamma_{\mp}^{(\mu,\sigma)}\left(\hat{\nabla}_{(\mu,\sigma)} \mp iTX'_{(\mu,\sigma)}\right)\\ \mathbf{D}_{K,\pm,\xi} &\equiv \mathbf{d}_{K,\xi} \pm \mathbf{d}^\dagger{}_{K,\xi}\,. \end{aligned} \qquad (6.57)$$

They are the supercharges which generate the super-Virasoro algebra in the usual chiral form

$$
\begin{aligned}
\{\mathbf{D}_{K,\pm,\xi_1}, \mathbf{D}_{K,\pm,\xi_2}\} &= 4\left(\pm\frac{1}{2}\mathbf{\Delta}_{\xi_1\xi_2} + i\mathcal{L}_{\xi_1\xi_2}\right) \\
\left[\pm\frac{1}{2}\mathbf{\Delta}_{K,\xi_1} + i\mathcal{L}_{\xi_1}, \mathbf{D}_{K,\pm,\xi_2}\right] &= 2\mathbf{D}_{K,\pm,\frac{1}{2}\xi_1'\xi_2 - \xi_1\xi_2'} \\
\left[\pm\frac{1}{2}\mathbf{\Delta}_{K,\xi_1} + i\mathcal{L}_{\xi_1}, \pm\frac{1}{2}\mathbf{\Delta}_{K,\xi_2} + i\mathcal{L}_{\xi_2}\right] &= 2i\left(\pm\frac{1}{2}\mathbf{\Delta}_{K,\xi_1'\xi_2 - \xi_1\xi_2'} + i\mathcal{L}_{\xi_1'\xi_2 - \xi_1\xi_2'}\right) .
\end{aligned}
\tag{6.58}
$$

It is easily seen that this acquires the standard form when we set $\xi(\sigma) = e^{in\sigma}$ for $n \in \mathbb{N}$. In order to make the connection with the usual formulation more transparent consider a flat target space. If we define the operators

$$
\mathcal{P}_{\pm,(\mu,\sigma)} \equiv \frac{1}{\sqrt{2T}}\left(-i\partial_{(\mu,\sigma)} \pm TX'_{(\mu,\sigma)}\right)
\tag{6.59}
$$

with commutator

$$
\left[\mathcal{P}_{A,(\mu,\sigma)}, \mathcal{P}_{B,(\nu,\sigma')}\right] = is_A\delta_{AB}\eta_{\mu\nu}\delta'_{\sigma,\sigma'} , \qquad \text{for } g_{\mu\nu} = \eta_{\mu\nu}
\tag{6.60}
$$

we get, up to a constant factor, the usual modes

$$
\begin{aligned}
\mathbf{D}_{K,\pm,\xi} &= \sqrt{2T}i\int d\sigma\,\xi(\sigma)\,\Gamma^{\mu}_{\mp}(\sigma)\,\mathcal{P}_{\mu,\mp}(\sigma) \\
\mathbf{D}^2_{K,\pm,\xi^2} &= \pm 2T\int d\sigma\,\left(\xi^2(\sigma)\,\mathcal{P}_{\mp}(\sigma)\cdot\mathcal{P}_{\mp}(\sigma) - \frac{i}{2}\xi(\sigma)\,(\xi\Gamma_{\mp})'(\sigma)\cdot\Gamma_{\mp}(\sigma)\right) .
\end{aligned}
\tag{6.61}
$$

6.3.2 Isomorphisms of the Superconformal Algebra

The representation of the superconformal algebra as above is manifestly of the form considered in §2.1.1 of [113]. We can therefore now study isomorphisms of the algebra along the lines of §2.1.2 of that paper in order to obtain new SCFTs from known ones.

From §2.1.2 of [113] it follows that the general continuous isomorphism of the 0-mode sector ($\xi = 1$) of the algebra (6.58) is induced by some operator

$$
\mathbf{W} = \int d\sigma\, W(\sigma) ,
\tag{6.62}
$$

where W is a field on loop space of unit reparameterization weight

$$
w(W) = 1 ,
\tag{6.63}
$$

and looks like

$$
\begin{aligned}
\mathbf{d}_{K,1} \mapsto \mathbf{d}^{\mathbf{W}}_{K,1} &\equiv \exp(-\mathbf{W})\,\mathbf{d}_{K,1}\exp(\mathbf{W}) \\
\mathbf{d^\dagger}_{K,1} \mapsto \mathbf{d^\dagger}^{\mathbf{W}}_{K,1} &\equiv \exp\left(\mathbf{W}^\dagger\right)\mathbf{d^\dagger}_{K,1}\exp\left(-\mathbf{W}^\dagger\right) \\
\mathbf{\Delta}_{K,1} \mapsto \mathbf{\Delta}^{\mathrm{W}}_{K,1} &\equiv \left\{\mathbf{d}^{\mathbf{W}}_{K,1}, \mathbf{d^\dagger}^{\mathbf{W}}_{K,1}\right\} \\
\mathcal{L}_1 \mapsto \mathcal{L}_1 .
\end{aligned}
\tag{6.64}
$$

This construction immediately generalizes to the full algebra including all modes

$$\begin{aligned}
\mathbf{d}_{K,\xi} &\mapsto \mathbf{d}^{\mathbf{W}}_{K,\xi} \;\equiv\; \exp(-\mathbf{W})\,\mathbf{d}_{K,\xi}\exp(\mathbf{W})\\
\mathbf{d}^{\dagger}{}_{K,\xi} &\mapsto \mathbf{d}^{\dagger}{}^{\mathbf{W}}_{K,\xi} \;\equiv\; \exp\Big(\mathbf{W}^{\dagger}\Big)\,\mathbf{d}^{\dagger}{}_{K,\xi}\exp\Big(-\mathbf{W}^{\dagger}\Big)\\
\mathcal{L}_{\xi} &\mapsto \mathcal{L}_{\xi}
\end{aligned} \tag{6.65}$$

if the crucial relation

$$\mathbf{\Delta}^{\mathbf{W}}_{K,\xi_1\xi_2} = \left\{\mathbf{d}^{\mathbf{W}}_{K,\xi_1}, \mathbf{d}^{\dagger}{}^{\mathbf{W}}_{K,\xi_2}\right\} \tag{6.66}$$

remains well defined, i.e. if (6.55) remains true:

$$\left\{\mathbf{d}^{\mathbf{W}}_{K,\xi_1\xi_3}, \mathbf{d}^{\dagger}{}^{\mathbf{W}}_{K,\xi_2}\right\} = \left\{\mathbf{d}^{\mathbf{W}}_{K,\xi_1}, \mathbf{d}^{\dagger}{}^{\mathbf{W}}_{K,\xi_2\xi_3}\right\}\,. \tag{6.67}$$

The form of these deformations follows from the fact that no matter which background fields are turned on, the generator (6.48) of spatial reparameterizations (at fixed worldsheet time) remains the same, because the string must be reparameterization invariant in any case. Preservation of the relation $\mathbf{d}_K^2 = i\mathcal{L}_K$, which says that $\mathbf{d}_K$ is nilpotent up to reparameterizations, then implies that $\mathbf{d}_K$ may transform under a similarity transformation as in the first line of (6.65). The rest of (6.65) then follows immediately.

Since this is an important point, at the heart of the approach presented here, we should also reformulate it in a more conventional language. Let L_m, $\bar{L}_m$, G_m, $\bar{G}_m$ be the holomorphic and antiholomorphic modes of the super Virasoro algebra. As discussed in §6.3.1 (p.104) we have

$$\begin{aligned}
\mathbf{\Delta}_{K,\xi} &\propto L_m + \bar{L}_{-m}\\
\mathcal{L}_{K,\xi} &\propto L_m - \bar{L}_{-m}\\
\mathbf{d}_{K,\xi} &\propto iG_m + \bar{G}_{-m}\\
\mathbf{d}^{\dagger}{}_{K,\xi} &\propto -iG_m + \bar{G}_{-m}\,,
\end{aligned} \tag{6.68}$$

with $\xi(\sigma) = e^{-im\sigma}$, as well as

$$\mathbf{W} \propto \sum_n W_n \bar{W}_n\,, \tag{6.69}$$

where W_m and $\bar{W}_m$ are the modes of the holomorphic and antiholomorphic parts of $\mathbf{W}$, which have weight h and $\bar{h}$, respectively. The goal is to find a deformation of (6.68) such that $L_m - \bar{L}_{-m}$ is preserved. Since this is the square of $\pm iG_m + \bar{G}_{-m}$ the latter may receive a similarity transformation which does not affect $L_m - \bar{L}_{-m}$ itself. Using $[L_m, W_n] = ((h-1)m - n)W_{n+m}$ and similarly for the antiholomorphic part we see that this is the case for

$$\begin{aligned}
iG_m + \bar{G}_{-m} &\to \exp\left(-\sum_n W_n\bar{W}_n\right)\left(iG_m + \bar{G}_{-m}\right)\exp\left(\sum_n W_n\bar{W}_n\right)\\
-iG_m + \bar{G}_{-m} &\to \exp\left(\sum_m \bar{W}_n^{\dagger}W_n^{\dagger}\right)\left(-iG_m + \bar{G}_{-m}\right)\exp\left(-\sum_n \bar{W}_n^{\dagger}W_n^{\dagger}\right)
\end{aligned} \tag{6.70}$$

with

$$h + \bar{h} = 1\,, \tag{6.71}$$

because then

$$L_m - \bar{L}_{-m} \to \exp\left(-\sum_n W_n \bar{W}_n\right) \left(L_m - \bar{L}_{-m}\right) \exp\left(\sum_n W_n \bar{W}_n\right) = L_m - L_{-m}\,. \tag{6.72}$$

The point of the loop-space formulation above is to clarify the nature of these deformations, which in terms of the $L_m, \bar{L}_m, G_m, \bar{G}_m$ look somewhat peculiar. In the loop space formulation it becomes manifest that we are dealing here with a generalization of the deformations first considered in [3] for supersymmetric quantum mechanics, where the supersymmetry generators are the exterior derivative and coderivative and are sent by two different similarity transformations to two new nilpotent supersymmetry generators. This and the relation to the present approach to superstrings is discussed in detail in section 2.1 of [113].

Every operator $\mathbf{W}$ which satisfies (6.63) and (6.66) hence induces a classical algebra isomorphism of the superconformal algebra (6.58). (Quantum corrections to these algebras can be computed and elimination of quantum anomalies will give background equations of motion, but this shall not be our concern here.) Finding such $\mathbf{W}$ is therefore a task analogous to finding superconformal Lagrangians in 2 dimensions.

However, two different $\mathbf{W}$ need not induce two different isomorphisms. In particular, *anti-Hermitean* $\mathbf{W}^\dagger = -\mathbf{W}$ induce *pure gauge* transformations in the sense that all algebra elements are transformed by the *same* unitary similarity transformation

$$\mathbf{X} \mapsto e^{-\mathbf{W}} \mathbf{X} e^{\mathbf{W}} \qquad \text{for } \mathbf{X} \in \left\{\mathbf{d}_{K,\xi}, \mathbf{d}^\dagger{}_{K,\xi}, \mathbf{\Delta}_{K,\xi}, \mathcal{L}_\xi\right\} \text{ and } \mathbf{W}^\dagger = -\mathbf{W}\,. \tag{6.73}$$

Examples for such unitary transformations are given in §6.3.3.4 (p.113) and §6.4.2 (p.121). They are related to background gauge transformations as well as to string dualities. For a detailed discussion of the role of such automorphism in the general framework of string duality symmetries see §7 of [123].

In the next subsections deformations of the above form are studied in general terms and by way of specific examples.

6.3.3 NS-NS Backgrounds

We start by deriving superconformal deformations corresponding to background fields in the NS-NS sector of the closed Type II string. Since the conformal weight of an NS-NS vertex comes from a *single* Wick contraction with the superconformal generators, while that of a spin field, which enters R-sector vertices, comes from a *double* Wick contraction, the deformation theory of NS-NS backgrounds is much more transparent than that of NS-R or R-NS sectors, as will be made clear in the following.

6.3.3.1 Gravitational background by algebra isomorphism. First we reconsider the purely gravitational background from the point of view that its superconformal algebra derives from the superconformal algebra for *flat* cartesian target space by a deformation of the form (6.65). For the point particle limit this was discussed in equations (38)-(42) of [113] and the generalization to loop space is straightforward: Denote by

$$\mathbf{d}^{\eta}_{K,1} \equiv \mathcal{E}^{\dagger(\mu,\sigma)}\partial_{(\mu,\sigma)} + i\mathcal{E}_{(\mu,\sigma)}X'^{(\mu,\sigma)} \tag{6.74}$$

the K-deformed exterior derivative on *flat* loop space and define the deformation operator by

$$\begin{aligned} \mathbf{W} &= \mathcal{E}^{\dagger}\cdot(\ln E)\cdot\mathcal{E} \\ &= \int d\sigma\; \mathcal{E}^{\dagger}(\sigma)\cdot(\ln e(X(\sigma)))\cdot\mathcal{E}\,, \end{aligned} \tag{6.75}$$

where $\ln E$ is the logarithm of a vielbein (6.17) on loop space, regarded as a matrix. This $\mathbf{W}$ is constructed so as to satisfy

$$e^{\mathbf{W}}\mathcal{E}^{\dagger a}(\sigma)\, e^{-\mathbf{W}} = \sum_{\nu} e^{a}{}_{\nu}\mathcal{E}^{\dagger(b=\nu)}\,, \tag{6.76}$$

which yields

$$\begin{aligned} e^{\mathbf{W}}\mathcal{E}^{\dagger\mu}(\sigma)\, e^{-\mathbf{W}} &= e^{\mathbf{W}}e^{\mu}{}_{a}\mathcal{E}^{\dagger a}(\sigma)\, e^{-\mathbf{W}} \\ &= e^{\mu}{}_{a}e^{a}{}_{\nu}\mathcal{E}^{\dagger(b=\nu)} \\ &= \mathcal{E}^{\dagger(b=\mu)}\,. \end{aligned} \tag{6.77}$$

Since $e^{\mathbf{W}}$ interchanges between two different vielbein fields which define two different metric tensors the index structure becomes a little awkward in the above equations. Since we won't need these transformations for the further developments we don't bother to introduce special notation to deal with this issue more cleanly. The point here is just to indicate that a $e^{\mathbf{W}}$ with the above properties does exist. It replaces all p-forms with respect to E by p-forms with respect to the flat metric. One can easily convince oneself that hence the operator $\mathbf{d}_K$ associated with the metric $G = E^2$ is related to the operator $\mathbf{d}^{\eta}_K$ for flat space by

$$\mathbf{d}_{K,\xi} = e^{-\mathbf{W}}\mathbf{d}^{\eta}_{K,\xi}e^{\mathbf{W}}\,. \tag{6.78}$$

Therefore, indeed, $\mathbf{W}$ of (6.75) induces a gravitational field on the target space.

As was discussed on p. 10 of [113] we need to require $\det e = 1$, and hence

$$\mathrm{tr}\ln e = 0 \tag{6.79}$$

in order that $\mathbf{d}^{\dagger\mathbf{W}}_{K,\xi} = (\mathbf{d}_{K,\xi^*})^{\dagger}$. This is just a condition on the nature of the coordinate system with respect to which the metric is constructed in our framework. As an abstract operator $\mathbf{d}_{K,\xi}$ is of course *independent* of any metric, its representation in terms of the operators $X^{(\mu,\sigma)}, \partial_{(\mu,\sigma)}, \mathcal{E}^{\dagger\mu}, \mathcal{E}^{\mu}$ is not, which is what the above is all about.

Note furthermore, that

$$\mathbf{W}^\dagger = \pm \mathbf{W} \Leftrightarrow (\ln e)^{\mathrm{T}} = \pm \ln e\,. \tag{6.80}$$

According to (6.73) this implies that the antisymmetric part of $\ln e$ generates a pure gauge transformation and *only the (traceless) symmetric part* of $\ln e$ is responsible for a perturbation of the gravitational background. A little reflection shows that the gauge transformation induced by antisymmetric $\ln e$ is a rotation of the vielbein frame. For further discussion of this point see pp. 58 of [128].

6.3.3.2 B-field background. As in §2.1.3 of [113] we now consider the Kalb-Ramond B-field 2-form

$$B = \frac{1}{2} B_{\mu\nu} dx^\mu \wedge dx^\nu \tag{6.81}$$

on target space with field strength $H = dB$. This induces on loop space the 2-form

$$B_{(\mu,\sigma)(\nu,\sigma')}(X) = B_{\mu\nu}(X(\sigma))\, \delta_{\sigma,\sigma'}\,. \tag{6.82}$$

We will study the deformation operator

$$\begin{aligned} \mathbf{W}^{(B)}(X) &\equiv \frac{1}{2} B_{(\mu,\sigma)(\nu,\sigma')}(X)\, \mathcal{E}^{\dagger(\mu,\sigma)} \mathcal{E}^{\dagger(\nu,\sigma')} \\ &\equiv \int d\sigma\, \frac{1}{2} B_{\mu\nu}(X(\sigma))\, \mathcal{E}^{\dagger\mu}(\sigma)\, \mathcal{E}^{\dagger\nu}(\sigma) \end{aligned} \tag{6.83}$$

on loop space (which is manifestly of reparameterization weight 1) and show that the superconformal algebra that it induces is indeed that found by a canonical treatment of the usual supersymmetric σ-model with gravitational and Kalb-Ramond background.

When calculating the deformations (6.65) explicitly for $\mathbf{W}$ as in (6.83) one finds

$$\begin{aligned} \mathbf{d}_{K,\xi}^{(B)} &\equiv \exp\left(-\mathbf{W}^{(B)}\right) \mathbf{d}_{K,\xi} \exp\left(\mathbf{W}^{(B)}\right) \\ &= \mathbf{d}_{K,\xi} + \left[\mathbf{d}_{K,\xi}, \mathbf{W}^{(b)}\right] \\ &= \int d\sigma\, \xi \left(\mathcal{E}^{\dagger\mu} \hat{\nabla}_\mu + iT\mathcal{E}_\mu X'^\mu + \frac{1}{6} H_{\alpha\beta\gamma}(X)\, \mathcal{E}^{\dagger\alpha}\mathcal{E}^{\dagger\beta}\mathcal{E}^{\dagger\gamma} - iT\mathcal{E}^{\dagger\mu} B_{\mu\nu}(X)\, X'^\nu\right) \\ \mathbf{d}^{\dagger(B)}_{K,\xi} &= \exp\left(\mathbf{W}^{\dagger(B)}\right) \mathbf{d}^\dagger{}_K \exp\left(-\mathbf{W}^{\dagger(B)}\right) \\ &= -\int d\sigma\, \xi(\sigma) \left(\mathcal{E}^\mu \hat{\nabla}_\mu + iT\mathcal{E}^\dagger_\mu X'^\mu + \frac{1}{6} H_{\alpha\beta\gamma}(X)\, \mathcal{E}^\alpha\mathcal{E}^\beta\mathcal{E}^\gamma - iT\mathcal{E}^\mu B_{\mu\nu}(X)\, X'^\nu\right)\,. \\ &. \end{aligned} \tag{6.84}$$

This is essentially equation (72) of [113], with the only difference that here we have mode functions ξ and an explicit realization of the deformation Killing vector.

In order to check that the above is a valid isomorphim condition (6.67) must be calculated. Concentrating on the potentially problematic terms one finds

$$\begin{aligned}
\left\{\mathbf{d}^{(B)}_{K,\xi_1},\mathbf{d}^{\dagger(B)}_{K,\xi_2}\right\} &= \int d\sigma\ \xi_1\xi_2(\cdots)\\
&\quad+\int d\sigma\, d\sigma'\xi_1(\sigma)\,\xi_2(\sigma')\, i\left(\mathcal{E}^\dagger_\mu(\sigma')-\mathcal{E}^\nu(\sigma')\,B_{\nu\mu}(X(\sigma'))\right)\mathcal{E}^{\dagger\mu}(\sigma)\,\delta'(\sigma',\sigma)\\
&\quad+\int d\sigma\, d\sigma'\xi_1(\sigma)\,\xi_2(\sigma')\, i\left(\mathcal{E}_\mu(\sigma)-\mathcal{E}^{\dagger\nu}(\sigma)\,B_{\nu\mu}(X(\sigma))\right)\mathcal{E}^\mu(\sigma')\,\delta'(\sigma,\sigma')\\
&= \int d\sigma\ \xi_1\xi_2(\cdots) - i\int d\sigma\ \left(\xi_1'\xi_2 B_{\nu\mu}\mathcal{E}^\nu\mathcal{E}^{\dagger\mu}+\xi_1\xi_2' B_{\nu\mu}\mathcal{E}^{\dagger\nu}\mathcal{E}^\mu\right)\\
&= \int d\sigma\ \xi_1\xi_2(\cdots)\,.
\end{aligned} \tag{6.85}$$

This expression therefore manifestly satisfies (6.67).

With hindsight this is no surprise, because (6.84) are precisely the superconformal generators in functional form as found by canonical analysis of the non-linear supersymmetric σ-model

$$S=\frac{T}{2}\int d^2\xi d^2\theta\ \left(G_{\mu\nu}+B_{\mu\nu}\right)D_+\mathbf{X}^\mu D_-\mathbf{X}^\nu\,, \tag{6.86}$$

where $\mathbf{X}^\mu$ are worldsheet superfields

$$\mathbf{X}^\mu(\xi,\theta_+,\theta_-)\equiv X^\mu(\xi)+i\theta_+\psi^\mu_+(\xi)-i\theta_-\psi^\mu_-(\xi)+i\theta_+\theta_-F^\mu(\xi)$$

and $D_\pm\equiv\partial_{\theta_\pm}-i\theta_\pm\partial_\pm$ with $\partial_\pm\equiv\partial_0\pm\partial_1$ are the superderivaties. The calculation can be found in section 2 of [10]. (In order to compare the final result, equations (32),(33) of [10], with our (6.84) note that our fermions $\Gamma_\pm$ are related to the fermions $\psi_\pm$ of [10] by $\Gamma_\pm=(i^{(1\mp1)/2}\sqrt{2T})\psi_\pm$.)

6.3.3.3 Dilaton background. The deformation operator in (6.75) which induces the gravitational background was of the form $\mathbf{W}=\mathcal{E}^\dagger\cdot M\cdot\mathcal{E}$ with M a traceless symmetric matrix. If instead we consider a deformation of the same form but for pure trace we get

$$\mathbf{W}^{(D)}=-\frac{1}{2}\int d\sigma\ \Phi(X)\,\mathcal{E}^{\dagger\mu}\mathcal{E}_\mu\,, \tag{6.87}$$

for some real scalar field Φ on target space. This should therefore induce a dilaton background. The associated superconformal generators are (we suppress the σ dependence and the mode functions ξ from now on)

$$\begin{aligned}
\exp\left(-\mathbf{W}^{(D)}\right)\mathbf{d}_K\exp\left(\mathbf{W}^{(D)}\right) &= e^{\Phi/2}\mathcal{E}^{\dagger\mu}\left(\hat\nabla_\mu-\frac{1}{2}(\partial_\mu\Phi)\mathcal{E}^{\dagger\nu}\mathcal{E}_\nu\right)+iTe^{-\Phi/2}X'^\mu\mathcal{E}_\mu\\
\exp\left(\mathbf{W}^{(D)}\right)\mathbf{d}^\dagger{}_K\exp\left(-\mathbf{W}^{(D)}\right) &= -e^{\Phi/2}\mathcal{E}^{\mu}\left(\hat\nabla_\mu+\frac{1}{2}(\partial_\mu\Phi)\mathcal{E}^{\dagger\nu}\mathcal{E}_\nu\right)-iTe^{-\Phi/2}X'^\mu\mathcal{E}^\dagger_\mu\,.
\end{aligned} \tag{6.88}$$

It is readily seen that for this deformation equation (6.67) is satisfied, so that these operators indeed generate a superconformal algebra.

The corresponding Dirac operators are

$$\mathbf{d}_K^{(\Phi)} \pm \mathbf{d}^\dagger{}_K^{(\Phi)} = \Gamma^\mu_\mp \left(e^{\Phi/2}\hat{\nabla}_\mu \mp iTe^{-\Phi/2} G_{\mu\nu} X'^\mu \right) \mp e^{\Phi/2}\Gamma^\mu_\pm (\partial_\mu \Phi)\mathcal{E}^{\dagger\nu}\mathcal{E}_\nu \,. \tag{6.89}$$

Comparison of the superpartners of $\Gamma_{\pm,\mu}$

$$\mp\frac{1}{2}\left\{\mathbf{d}_K^{(\Phi)} \pm \mathbf{d}^\dagger{}_K^{(\Phi)}, \Gamma_{\mp,\mu}\right\} = e^{\Phi/2}\partial_\mu \mp iTe^{-\Phi/2}G_{\mu\nu}X'^\mu + \text{fermionic terms} \tag{6.90}$$

with equation (6.192) in appendix §6.5 (p.130) shows that this has the form expected for the dilaton coupling of a D-string.

6.3.3.4 Gauge field background. A gauge field background $A = A_\mu dx^\mu$ should express itself via $B \to B - \frac{1}{T}F$, where $F = dA$ (e.g. §8.7 of [129]), if we assume A to be a $U(1)$ connection for the moment. Since the present discussion so far refers only to closed strings and since closed strings have trivial coupling to A it is to be expected that an A-field background manifests itself as a pure gauge transformation in the present context. This motivates to investigate the deformation induced by the anti-Hermitean

$$\mathbf{W} = iA_{(\mu,\sigma)}(X)\, X'^{(\mu,\sigma)} \;=\; i\int d\sigma\; A_\mu(X(\sigma))\, X'^\mu(\sigma) \;. \tag{6.91}$$

The associated superconformal generators are found to be

$$\begin{aligned} \mathbf{d}_K^{(A)(B)} &= \mathbf{d}_K^{(B)} + i\mathcal{E}^{\dagger\mu}F_{\mu\nu}X'^\nu \\ \mathbf{d}^\dagger{}_K^{(A)(B)} &= \mathbf{d}^\dagger{}_K^{(B)} - i\mathcal{E}^{\mu}F_{\mu\nu}X'^\nu \,. \end{aligned} \tag{6.92}$$

Comparison with (6.84) shows that indeed

$$\mathbf{d}_K^{(A)(B)} = \mathbf{d}_K^{(B-\frac{1}{T}F)} \,, \tag{6.93}$$

so that we can identify the background induced by (6.91) with that of the NS $U(1)$ gauge field.

Since $\exp(\mathbf{W})(X)$ is nothing but the Wilson loop of A around X, it is natural to conjecture that for a general (non-abelian) gauge field background A the corresponding deformation is the Wilson loop as well:

$$\mathbf{d}_K^{(A)} = \left(\mathrm{Tr}\mathcal{P}e^{-i\int A_\mu X'^\mu}\right)\mathbf{d}_K\left(\mathrm{Tr}\mathcal{P}e^{+i\int A_\mu X'^\mu}\right) \,, \tag{6.94}$$

where $\mathcal{P}$ indicates path ordering and Tr the trace in the Lie algebra, as usual.

6.3.3.5 C-field background. So far we have found deformation operators for all massless NS and NS-NS background fields. One notes a close similarity between the form of these deformation operators and the form of the corresponding vertex operators (in fact, the deformation operators are related to the vertex operators in the (-1,-1) picture. This is discussed in §6.3.4.2 (p.117)): The deformation operators for G, B and Φ are bilinear in the form creation/annihilation operators on loop space, with the bilinear form (matrix) seperated into its traceless symmetric, antisymmetric and trace part.

Interestingly, though, there is one more deformation operator obtainable by such a bilinear in the form creation/annihilation operators. It is

$$\mathbf{W}^{(C)} \equiv \frac{1}{2}\int d\sigma\; C_{\mu\nu}(X)\,\mathcal{E}^\mu\mathcal{E}^\nu\,, \tag{6.95}$$

i.e. the *adjoint* of (6.83). It induces the generators

$$\begin{aligned}
\mathbf{d}^{(C)}_{K,\xi} &= \int d\sigma\;\xi\Big(\mathcal{E}^{\dagger\mu}\hat{\nabla}_\mu + i\mathcal{E}_\mu X'^\mu - \mathcal{E}^\nu C_\nu{}^\mu\hat{\nabla}_\mu + \frac{1}{2}\mathcal{E}^{\dagger\alpha}\mathcal{E}^\mu\mathcal{E}^\nu(\nabla_\alpha C_{\mu\nu})\\
&\qquad -\frac{1}{2}C_\nu{}^\mu\mathcal{E}^\nu\mathcal{E}^\alpha\mathcal{E}^\beta(\nabla_\mu C_{\alpha\beta}) + \frac{1}{2}C^\alpha{}_\beta\mathcal{E}^\beta\mathcal{E}^\mu\mathcal{E}^\nu(\nabla_\alpha C_{\mu\nu})\Big)\\
\mathbf{d}^{\dagger(C)}_{K,\xi} &= -\int d\sigma\;\xi\Big(\mathcal{E}^{\mu}\hat{\nabla}_\mu + i\mathcal{E}^\dagger_\mu X'^\mu - \mathcal{E}^{\dagger\nu} C_\nu{}^\mu\hat{\nabla}_\mu + \frac{1}{2}\mathcal{E}^{\alpha}\mathcal{E}^{\dagger\mu}\mathcal{E}^{\dagger\nu}(\nabla_\alpha C_{\mu\nu})\\
&\qquad -\frac{1}{2}C_\nu{}^\mu\mathcal{E}^{\dagger\nu}\mathcal{E}^{\dagger\alpha}\mathcal{E}^{\dagger\beta}(\nabla_\mu C_{\alpha\beta}) + \frac{1}{2}C^\alpha{}_\beta\mathcal{E}^{\dagger\beta}\mathcal{E}^{\dagger\mu}\mathcal{E}^{\dagger\nu}(\nabla_\alpha C_{\mu\nu})\Big)\,.
\end{aligned} \tag{6.96}$$

Furthermore it turns out that this deformation, too, does respect (6.67): When we again concentrate only on the potentially problematic terms we see that

$$\begin{aligned}
\left\{\mathbf{d}^{(C)}_{K,\xi_1},\mathbf{d}^{\dagger(C)}_{K,\xi_2}\right\} &= \int d\sigma\;\xi_1\xi_2(\cdots)\\
&\quad + \left\{-\int d\sigma\;\xi_1\mathcal{E}^\nu C_\nu{}^\mu\hat{\nabla}_\mu, -i\int d\sigma\;\xi_2\mathcal{E}^\dagger_\mu X'^\mu\right\}\\
&\quad + \left\{\int d\sigma\;\xi_2\mathcal{E}^{\dagger\nu} C_\nu{}^\mu\hat{\nabla}_\mu, i\int d\sigma\;\xi_1\mathcal{E}_\mu X'^\mu\right\}\\
&= \int d\sigma\;\xi_1\xi_2\,(\cdots)\\
&\quad + i\int d\sigma\;\left(\xi_1\xi_2'\;\mathcal{E}^\dagger_\mu C_\nu{}^\mu\mathcal{E}^\nu + \xi_1'\xi_2\;\mathcal{E}_\mu C_\nu{}^\mu\mathcal{E}^{\dagger\nu}\right)\\
&= \int d\sigma\;\xi_1\xi_2\,(\cdots)\,.
\end{aligned} \tag{6.97}$$

Therefore (6.96) do generate a superconformal algebra and hence define an SCFT.

What, though, is the physical interpretation of the field C on spacetime? It is apparently not the NS 2-form field, because the generators (6.96) are different from (6.84) and don't seem to be unitarily equivalent. A possible guess would therefore be that it is the *RR 2-form* C_2, but now coupled to a D-string instead of an F-string.

The description of the F-string in an RR background would involve ghosts and spin fields, which we do not discuss here. But the coupling of the D-string to the RR 2-form is very similar to the coupling of the F-string to the Kalb-Ramond 2-form and does not involve any spin fields. That's why the above deformation might allow an interpretation in terms of D-strings in RR 2-form backgrounds.

But this needs to be further examined. A hint in this direction is that under a duality transformation which changes the sign of the dilaton, the C-field is exchanged with the B-field. This is discussed in §6.4.2.2 (p.125).

6.3.4 Canonical Deformations and Vertex Operators

With all NS-NS backgrounds under control (§6.3.3 (p.109)) we now turn to a more general analysis of the deformations of §6.3.2 (p.107) that puts the results of the previous subsections in perspective and shows how general backgrounds are to be handled.

6.3.4.1 Review of first order canonical CFT deformations. Investigations of conformal deformations by way of adding terms to the conformal generators go back as far as[4] [119], which builds on earlier insights [130, 131] into continuous families of conformal algebras.

It has been noted long ago [116] that adding an integrated background vertex operator V (a worldsheet field of weight (1,1)) to the string's action to first order induces a perturbation

$$L_m \to L_m + \int d\sigma\, e^{-im\sigma} V(\sigma) \tag{6.98}$$

of the Virasoro generators and a similar shift occurs for the supercurrent [115].

While in [116] this is discussed in CFT language it becomes quite transparent in canonical language: From the string's worldsheet action for gravitational $G_{\mu\nu}$, Kalb-Ramond $B_{\mu\nu}$ and dilaton Φ background one finds the classical stress-energy tensor (*cf.* §6.5 (p.130))

$$T(\sigma) = \frac{1}{2} G^{\mu\nu} \frac{1}{\sqrt{2T}} \left(e^{\Phi/2} P_\mu + T\left(e^{\Phi/2} B_{\mu\kappa} + e^{-\Phi/2} G_{\mu\kappa}\right) X'^\kappa\right) \frac{1}{\sqrt{2T}} \left(e^{\Phi/2} P_\nu + T\left(e^{\Phi/2} B_{\nu\kappa} + e^{-\Phi/2} G_{\nu\kappa}\right) X'^\kappa\right)(\sigma)\ , \tag{6.99}$$

where P_μ is the canonical momentum to X^μ.

When expanded in terms of small perturbations

$$\begin{aligned} G_{\mu\nu}(X) &= \eta_{\mu\nu} + h_{\mu\nu}(X) + \cdots \\ B_{\mu\nu}(X) &= 0 + b_{\mu\nu}(X) + \cdots \\ \Phi(X) &= 0 + \phi(X) + \cdots \end{aligned} \tag{6.100}$$

of the background fields this yields

$$\begin{aligned} T &\approx \frac{1}{2}\left(\eta^{\mu\nu} - h^{\mu\nu}\right)\left(\mathcal{P}_{+\mu} + \sqrt{\frac{T}{2}} b_{\mu\kappa} X'^\kappa + \sqrt{\frac{T}{2}} h_{\mu\kappa} X'^\kappa + \frac{1}{\sqrt{8T}}\phi P_\mu - \sqrt{\frac{T}{8}}\phi\eta_{\mu\kappa} X'^\kappa\right)\left(\cdots\right)_\nu \\ &= \frac{1}{2}\eta_{\mu\nu}\mathcal{P}_+^\mu\mathcal{P}_+^\nu - \underbrace{\frac{1}{2}h_{\mu\nu}\mathcal{P}_+^\mu\mathcal{P}_-^\nu}_{\equiv V_G} - \underbrace{\frac{1}{2}b_{\mu\nu}\mathcal{P}_+^\mu\mathcal{P}_-^\nu}_{\equiv V_B} + \underbrace{\frac{1}{2}\phi\eta_{\mu\nu}\mathcal{P}_+^\mu\mathcal{P}_-^\nu}_{\equiv V_\Phi} + \text{higher order terms}\,, \end{aligned} \tag{6.101}$$

where we have defined

$$\mathcal{P}_\pm^\mu(\sigma) \equiv \frac{1}{\sqrt{2T}}\left(\eta^{\mu\nu} P_\nu \pm T X'^\nu\right)(\sigma)\ . \tag{6.102}$$

[4] We are grateful to M. Halpern for making us aware of this work.

It must be noted that while the objects $\mathcal{P}_\pm$, which have Poisson-bracket

$$\{\mathcal{P}^\mu_\pm(\sigma)\,,\mathcal{P}^\nu_\pm(\sigma')\} = \mp\eta^{\mu\nu}\delta'(\sigma-\sigma')\ , \tag{6.103}$$

generate the current algebra of the free theory, they involve, via $P_\mu = \delta S/\delta\dot{X}^\mu$, data of the perturbed background and are hence not proportional to ∂X and $\bar{\partial}X$.

Still, the first term in (6.101) is the generator of the Virasoro algebra which is associated with the U(1)-currents $\mathcal{P}_\pm$, while the following terms are the weight (1,1) vertices V_G, V_B, V_Φ (with respect to the first term) of the graviton, 2-form and dilaton, respectively.

Hence in the sense that we regard the canonical coordinates and momenta as fundamental and hence unaffected by the background perturbation, i.e.

$$\begin{aligned} X^\mu &\to X^\mu \\ P_\mu &\to P_\mu\,, \end{aligned} \tag{6.104}$$

while only the 'coupling constants' are shifted

$$\eta_{\mu\nu} \to \eta_{\mu\nu} + h_{\mu\nu}\,, \qquad \text{etc.} \tag{6.105}$$

we can write

$$T \to T + V\,, \tag{6.106}$$

where V denotes a collection of weight (1,1) vertices in the above sense.[5]

CFT deformations of this form are called *canonical deformations* [117, 132].

The central idea of canonical first order deformations is that the (super-) Virasoro algebra

$$\begin{aligned} \big[T(\sigma),T(\sigma')\big] &= 2iT(\sigma')\,\delta'(\sigma-\sigma') - iT'(\sigma')\,\delta(\sigma-\sigma') + A(\sigma-\sigma') \\ \big\{T_F(\sigma),T_F(\sigma')\big\} &= -\frac{1}{2\sqrt{2}}T(\sigma')\,\delta(\sigma') + B(\sigma-\sigma') \\ \big[T(\sigma),T_F(\sigma')\big] &= \frac{3i}{2}T_F(\sigma')\,\delta'(\sigma-\sigma') - iT_F'(\sigma')\,\delta(\sigma-\sigma') \end{aligned} \tag{6.107}$$

(where A and B are the anomaly terms) together with its chiral partner, generated by $\bar{T}$ and $\bar{T}_F$, is preserved to first order under the perturbation

$$\begin{aligned} T(\sigma) &\to T(\sigma) + \delta T(\sigma) \\ T_F(\sigma) &\to T_F(\sigma) + \delta T_F(\sigma) \end{aligned} \tag{6.108}$$

if, in particular,

$$\begin{aligned} \delta T(\sigma) &= \Phi(\sigma)\,\bar{\Phi}(\sigma) \\ \delta F(\sigma) &= \Phi_F(\sigma)\,\bar{\Phi}_F(\sigma) \end{aligned} \tag{6.109}$$

[5] As is discussed in [116], the issue concerning (6.104) in CFT language translates into the question whether one chooses to treat ∂X and $\bar{\partial}X$ as free fields in the perturbed theory and whether the $\partial X\,\partial X$-OPE is taken to receive a perturbation or not.

For a further discussion of perturbations of SCFTs where this issue is addressed, see [113] and in particular section 2.2.4.

with

$$
\begin{aligned}
\left[T(\sigma),\Phi(\sigma')\right] &= i\Phi(\sigma')\,\delta'(\sigma-\sigma') - i\Phi'(\sigma')\,\delta(\sigma-\sigma')\\
\left[T(\sigma),\Phi_F(\sigma')\right] &= \frac{i}{2}\Phi(\sigma')\,\delta'(\sigma-\sigma') - i\Phi'(\sigma')\,\delta(\sigma-\sigma')\\
\left[T(\sigma),\bar\Phi(\sigma')\right] &= 0\\
\left[T(\sigma),\bar\Phi_F(\sigma')\right] &= 0
\end{aligned}
\tag{6.110}
$$

and analogous relations for $\delta\bar T$ and $\delta\bar T_F$.

There are however also more general fields δT, δT_F of total weight 2 and 3/2, respectively, which preserve the above super-Virasoro algebra to first order [133]. But the weight $(1,1)$ part $\Phi(\sigma)\,\bar\Phi(\sigma)$ is special in that it corresponds directly to the vertex operator of the background which is described by the deformed worldsheet theory. Further deformation fields of weight different from (1,1) are related to *gauge* degrees of freedom of the background fields (*cf.* [133] and the discussion below equation (6.121)).

6.3.4.2 Canonical deformations from $\mathbf{d}_K \to e^{-\mathbf{W}}\mathbf{d}_K e^{\mathbf{W}}$. We would like to see how the deformation theory reviewed above relates to the SCFT deformations that have been studied in §6.3 (p.104).

First recall from §6.3.1 (p.104) that the chiral bosonic fields in our notation read

$$
\mathcal{P}_\pm(\sigma) \equiv \frac{1}{\sqrt{2T}}\left(-i\frac{\delta}{\delta X} \pm TX'\right)(\sigma)
\tag{6.111}
$$

and that according to §6.2.2.2 (p.101) we write the worldsheet fermions ψ, $\bar\psi$ as $\Gamma_\pm$, respectively, which are normalized so that $\left\{\Gamma^\mu_\pm(\sigma),\Gamma^\nu_\pm(\sigma')\right\} = \pm 2g^{\mu\nu}(X(\sigma))\,\delta(\sigma-\sigma')$, and we frequently make use of the linear combinations

$$
\begin{aligned}
\mathcal{E}^{\dagger\mu} &= \frac{1}{2}\left(\Gamma^\mu_+ + \Gamma^\mu_-\right)\\
\mathcal{E}^{\mu} &= \frac{1}{2}\left(\Gamma^\mu_+ - \Gamma^\mu_-\right)\,.
\end{aligned}
\tag{6.112}
$$

In this notation the supercurrents for the trivial background read

$$
\begin{aligned}
T_F(\sigma) &= \frac{1}{\sqrt{2}}\Gamma_+(\sigma)\,\mathcal{P}_+(\sigma) \;=\; \frac{-i}{\sqrt{4T}}\Big(\mathbf{d}_K(\sigma) - \mathbf{d}^\dagger{}_K(\sigma)\Big)\\
\bar T_F(\sigma) &= \frac{i}{\sqrt{2}}\Gamma_-(\sigma)\,\mathcal{P}_-(\sigma) \;=\; \frac{1}{\sqrt{4T}}\Big(\mathbf{d}_K(\sigma) + \mathbf{d}^\dagger{}_K(\sigma)\Big)\,,
\end{aligned}
\tag{6.113}
$$

where the K-deformed exterior derivative and coderivative on loop space are identified as

$$
\begin{aligned}
\mathbf{d}_K &= \sqrt{T}\left(\bar T_F + iT_F\right)\\
\mathbf{d}^\dagger{}_K &= \sqrt{T}\left(\bar T_F - iT_F\right)\,.
\end{aligned}
\tag{6.114}
$$

According to §6.3.2 (p.107) a consistent deformation of the superconformal algebra generated by T_F and $\bar{T}_F$ is given by sending

$$\mathbf{d}_K(\sigma) \;\to\; \mathbf{d}_K^{(W)}(\sigma) = e^{-\mathbf{W}}\mathbf{d}_K(\sigma)\, e^{\mathbf{W}} = \mathbf{d}_K(\sigma) + \underbrace{[\mathbf{d}_K(\sigma), \mathbf{W}]}_{\equiv \delta \mathbf{d}_K(\sigma)} + \cdots$$

$$\mathbf{d}^\dagger{}_K(\sigma) \;\to\; \mathbf{d}^\dagger{}_K^{(W)}(\sigma) = e^{\mathbf{W}^\dagger}\mathbf{d}^\dagger{}_K(\sigma)\, e^{-\mathbf{W}^\dagger} = \mathbf{d}^\dagger{}_K(\sigma) + \underbrace{\left[\mathbf{W}^\dagger, \mathbf{d}^\dagger{}_K(\sigma)\right]}_{\equiv \delta \mathbf{d}^\dagger{}_k(\sigma)} + \cdots \qquad (6.115)$$

for $\mathbf{W}$ some reparameterization invariant operator. From this one finds δT_F by using (6.113)

$$\delta T_F = -i\left[\bar{T}_F, \frac{1}{2}\left(\mathbf{W} + \mathbf{W}^\dagger\right)\right] + \left[T_F, \frac{1}{2}\left(\mathbf{W} - \mathbf{W}^\dagger\right)\right]$$

$$\delta \bar{T}_F = i\left[T_F, \frac{1}{2}\left(\mathbf{W} + \mathbf{W}^\dagger\right)\right] + \left[\bar{T}_F, \frac{1}{2}\left(\mathbf{W} - \mathbf{W}^\dagger\right)\right] \qquad (6.116)$$

which again gives δT by means of

$$\left\{T_F(\sigma), \delta T_F\left(\sigma'\right)\right\} + \left\{T_F\left(\sigma'\right), \delta T_F(\sigma)\right\} = -\frac{1}{2\sqrt{2}}\delta T(\sigma)\, \delta\left(\sigma - \sigma'\right) . \qquad (6.117)$$

Before looking at special cases one should note that this necessarily implies that δT_F is of total weight 3/2 and that δT is of total weight 2. That is because $\mathbf{W}$, being reparameterization invariant, must be the integral (along the string at fixed worldsheet time) over a field of unit total weight (*cf.* (6.63) and (6.71)) and because supercommutation with $\mathbf{d}_K$ or $\mathbf{d}^\dagger{}_K$ increases the total weight by 1/2.

Furthermore, recall from (6.73) that the anti-hermitean part $\frac{1}{2}\left(\mathbf{W} - \mathbf{W}^\dagger\right)$ of the deformation operator $\mathbf{W}$ is responsible for *pure gauge transformations* while the hermitean part $\frac{1}{2}\left(\mathbf{W} + \mathbf{W}^\dagger\right)$ induces true modifications of the background fields. Hence for a pure gauge transformation (6.116) yields

$$\delta T_F = [T_F, \mathbf{W}]$$

$$\delta \bar{T}_F = \left[\bar{T}_F, \mathbf{W}\right] , \qquad \text{for } \mathbf{W}^\dagger = -\mathbf{W}^\dagger , \qquad (6.118)$$

which of course comes from the global similarity transformation (6.73)

$$\mathbf{X} \mapsto e^{-\mathbf{W}}\mathbf{X}e^{\mathbf{W}} , \qquad \mathbf{X} \in \left\{T_F, \bar{T}_F, \cdots\right\} . \qquad (6.119)$$

On the other hand, for a strictly non-gauge transformation the transformation (6.116) simplifies to

$$\delta T_F = -i\left[\bar{T}_F, \mathbf{W}\right]$$

$$\delta \bar{T}_F = i\left[T_F, \mathbf{W}\right] , \qquad \text{for } \mathbf{W}^\dagger = +\mathbf{W} . \qquad (6.120)$$

In the cases where $\mathbf{W}$ is antihermitean *and* a $(1/2, 1/2)$ field (as is in particular the case for the gravitational $\mathbf{W}^{(G)}$ of §6.3.3.1 (p.110), the dilaton $\mathbf{W}^{(D)}$ of §6.3.3.3 (p.112) and

the hermitean part of the Kalb-Ramond $\mathbf{W}^{(B)} + \mathbf{W}^{(B)\dagger}$ of §6.3.3.2 (p.111)) this, together with (6.117) implies that

$$\delta T \propto \left\{T_F, \left[\bar{T}_F, \mathbf{W}\right]\right\} \tag{6.121}$$

is indeed of weight $(1,1)$, as discussed in the theory of canonical deformations §6.3.4.1 (p.115). Furthermore, this shows explicitly that all contributions to δT which are of total weight 2 but *not* of weight (1,1) must come from the antihermitean component $\frac{1}{2}\left(\mathbf{W} - \mathbf{W}^\dagger\right)$ and hence must be associated with background gauge transformations. (This proves in full generality the respective observation in [133] concerning 2-form field deformations.)

Finally, equation (6.121) clarifies exactly how the deformation operators $\mathbf{W}$ are related to the *vertex operators* of the respective background fields, namely it shows that the hermitean part of $\mathbf{W}$ is proportional to the vertex operator in the (-1,-1) picture (i.e. the pre-image under $\left\{T_F, \left[\bar{T}_F, \cdot\right]\right\}$).

As an example, consider the deformation induced by a B-field background:
According to §6.3.3.2 (p.111) a Kalb-Ramond background is induced by choosing

$$\mathbf{W} = \int d\sigma\, \frac{1}{2} B_{\mu\nu}(X(\sigma))\, \mathcal{E}^{\dagger\mu}\mathcal{E}^{\dagger\nu} \tag{6.122}$$

which, using (6.115) gives rise to

$$\begin{aligned} \delta\mathbf{d}_K(\sigma) &= \left(\frac{1}{6}H_{\alpha\beta\gamma}(X)\, \mathcal{E}^{\dagger\alpha}\mathcal{E}^{\dagger\beta}\mathcal{E}^{\dagger\gamma} - iT\mathcal{E}^{\dagger\mu}B_{\mu\nu}(X)\, X'^{\nu}\right)(\sigma) \\ \delta\mathbf{d}^\dagger{}_K(\sigma) &= \left(-\frac{1}{6}H_{\alpha\beta\gamma}(X)\, \mathcal{E}^{\alpha}\mathcal{E}^{\beta}\mathcal{E}^{\gamma} + iT\mathcal{E}^{\mu}B_{\mu\nu}(X)\, X'^{\nu}\right)(\sigma) \end{aligned} \tag{6.123}$$

and hence, using (6.113), to

$$\delta T_F(\sigma) = -\frac{i}{12\sqrt{T}}H_{\alpha\beta\gamma}\left(\mathcal{E}^{\dagger\alpha}\mathcal{E}^{\dagger\beta}\mathcal{E}^{\dagger\gamma} + \mathcal{E}^{\alpha}\mathcal{E}^{\beta}\mathcal{E}^{\gamma}\right) - \frac{1}{\sqrt{8}}\Gamma^\mu_+ B_{\mu\nu}\left(\mathcal{P}^\nu_+ - \mathcal{P}^\nu_-\right)\,. \tag{6.124}$$

In this special case δT_F happens to be the exact shift of T_F (there are no higher order perturbations of T_F in this background). As has been noted already in §6.3.3.2 (p.111) the same result is obtained by canonically quantizing the supersymmetric $2d$ σ-model (6.86) which describes superstrings in a Kalb-Ramond background.

By means of (6.117) the shift δT is easily found to be

$$\begin{aligned} \delta T(\sigma) = \Bigg(&-\frac{1}{12T}\partial_\delta H_{\alpha\beta\gamma}\left(\mathcal{E}^{\delta}\mathcal{E}^{\dagger\alpha}\mathcal{E}^{\dagger\beta}\mathcal{E}^{\dagger\gamma} + \mathcal{E}^{\dagger\delta}\mathcal{E}^{\alpha}\mathcal{E}^{\beta}\mathcal{E}^{\gamma}\right) - i\frac{1}{2\sqrt{2T}}H_{\alpha\beta\gamma}\left(\mathcal{E}^{\dagger\alpha}\mathcal{E}^{\dagger\beta} + \mathcal{E}^{\alpha}\mathcal{E}^{\beta}\right)\mathcal{P}^\gamma_+ \\ &+\frac{i}{\sqrt{4T}}\partial_\delta B_{\mu\nu}\left(\mathcal{P}^\nu_+ - \mathcal{P}^\nu_-\right)\Gamma^\delta_+\Gamma^\mu_+ + B_{\mu\nu}\mathcal{P}^\mu_+\mathcal{P}^\nu_-\Bigg)(\sigma)\,. \end{aligned} \tag{6.125}$$

This is of total weight 2 and contains the weight (1,1) vertex operator

$$V = B_{\mu\nu}\mathcal{P}^\mu_+\mathcal{P}^\nu_- \tag{6.126}$$

of the Kalb-Ramond field (*cf.* eqs. (52),(53) in [133]). That $T + \delta T$ satisfies the Virasoro algebra to first order at the level of Poisson brackets follows from the fact that it derives from a consistent deformation of the form (6.65) (as well as from the fact that it also derives from the respective σ-model Lagrangian).

6.4 Relations between the various Superconformal Algebras

We have found classical deformations of the superconformal algebra associated with several massless target space background fields. The special algebraic nature of the form in which we obtain these superconformal algebras admits a convenient treatment of gauge and duality transformations among the associated background fields. This is discussed in the following.

6.4.1 $\mathbf{d}_K$-exact Deformation Operators

Deformation operators $\mathbf{W}$ which are $\mathbf{d}_K$-exact, i.e. which are of the form

$$\mathbf{W} = \left[\mathbf{d}_K, \mathbf{w}\right]_\iota \,, \tag{6.127}$$

(where $[\cdot,\cdot]_\iota$ is the supercommutator) and which furthermore satisfy

$$\mathbf{W} = \left[\mathbf{d}_{K,\xi}, \mathbf{w}_{\xi^{-1}}\right]_\iota \,, \qquad \forall\, \xi \tag{6.128}$$

(where $\mathbf{w}_\xi \equiv \int d\sigma\; \xi\; w(\sigma)$) are special because for them[6]

$$\left[\mathbf{d}_{K,\xi}, \left[\mathbf{d}_K, \mathbf{w}\right]_\iota\right]_\iota = 0 \tag{6.130}$$

and hence they leave the generators of the algebra (6.58) invariant:

$$\mathbf{d}^{\mathbf{W}}_{K,\xi} = \mathbf{d}_{K,\xi} \,. \tag{6.131}$$

Two interesting choices for $\mathbf{w}$ are

$$\mathbf{w} = A_{(\mu,\sigma)}(X)\, \mathcal{E}^{\dagger(\mu,\sigma)} \tag{6.132}$$

and

$$\mathbf{w} = V^{(\mu,\sigma)}(X)\, \mathcal{E}_{(\mu,\sigma)} \,, \tag{6.133}$$

which both satisfy (6.128). They correspond to B-field gauge transformations and to diffeomorphisms, respectively:

[6] One way to see this is the following:

$$\begin{aligned}
\left[\mathbf{d}_{K,\xi}, \left[\mathbf{d}_K, \mathbf{w}\right]_\iota\right]_\iota &= \left[\mathbf{d}_{K,\xi}, \left[\mathbf{d}_{K,\xi}, \mathbf{w}_{\xi^{-1}}\right]_\iota\right]_\iota \\
&= \left[\mathcal{L}_{K,\xi^2}, \mathbf{w}_{\xi^{-1}}\right]_\iota \\
&= \int d\sigma\; \left(\xi^2\xi^{-1}w' + \frac{1}{2}(\xi^2)'\xi^{-1}w\right)(\sigma) \\
&= \int d\sigma\; (\xi w)' \,,
\end{aligned} \tag{6.129}$$

where we used that $w(\sigma)$ must be of weight 1/2 in order that $W(\sigma)$ satisfies condition (6.63).

6.4.1.1 B-field gauge transformations. For the choice (6.132) one gets

$$\begin{aligned}\mathbf{W} &= \left\{\mathbf{d}_K, A_{(\mu,\sigma)}\mathcal{E}^{\dagger(\mu,\sigma)}\right\} \\ &= \frac{1}{2}(dA)_{(\mu,\sigma)(\nu,\sigma')}\mathcal{E}^{\dagger(\mu,\sigma)}\mathcal{E}^{\dagger(\nu,\sigma')} + iTA_{(\mu,\sigma)}X'^{(\mu,\sigma)} \,. \end{aligned} \tag{6.134}$$

Comparison with (6.83) and (6.91) shows that this $\mathbf{W}$ induces a B-field background with $B = dA$ and a gauge field background with $F = T\,dA$. According to (6.93) these two backgrounds indeed precisely cancel.

This ties up a loose end from §6.3.3.2 (p.111): A pure gauge transformation $B \to B+dA$ of the B-field does not affect physics of the closed string and hence should manifest itself as an algebra isomorphism. Indeed, this isomorphism is that induced by $\mathbf{W} = iTA_{(\mu,\sigma)}X'^{(\mu,\sigma)}$.

6.4.1.2 Target space diffeomorphisms. For the choice (6.133) one gets

$$\begin{aligned}\mathbf{W} &= \left\{\mathbf{d}_K, V^{(\mu,\sigma)}\mathcal{E}_{(\mu,\sigma)}\right\} \\ &= \int d\sigma\; \left(V^\mu\partial_\mu + (\partial_\mu V^\nu)\mathcal{E}^{\dagger\mu}\mathcal{E}_\nu\right)(\sigma) \\ &= \mathcal{L}_V \,, \end{aligned} \tag{6.135}$$

where $\mathcal{L}_V$ is the operator inducing the Lie derivative along V on forms over loop space (*cf.* A.4 of [113]). According to §6.3.3.1 (p.110) the part involving $(\partial_\mu V^\nu)\mathcal{E}^{\dagger\mu}\mathcal{E}_\nu$ changes the metric field at every point of target space, while the part involving $V^\mu\partial_\mu$ translates the fields that enter in the superconformal generators. This $\mathbf{W}$ apparently induces target space diffeomorphisms.

6.4.2 T-duality

It is well known ([8] and references given there) that in the context of the non-commutative-geometry description of stringy spacetime physics T-duality corresponds to an inner automorphism

$$\mathcal{T} : \mathcal{A} \to e^{-\mathbf{W}}\,\mathcal{A}\,e^{\mathbf{W}} = \mathcal{A} \qquad \text{with } \mathbf{W}^\dagger = -\mathbf{W} \tag{6.136}$$

of the algebra $\mathcal{A}$ that enters the spectral triple. This has been worked out in detail for the bosonic string in [124]. In the following this construction is adapted to and rederived in the present context for the superstring and then generalized to the various backgrounds that we have found by deformations.

Following [8] we first consider T-duality along all dimensions, or equivalently, restrict attention to the field components along the directions that are T-dualized. Then we show that the *Buscher rules* (see [134] for a recent reference) for factorized T-duality (i.e. for T-duality along only a single direction) can very conveniently be derived in our framework, too.

6.4.2.1 Ordinary T-duality. Since T-dualizing along spacetime directions that are not characterized by commuting isometries is a little subtle (*cf.* §4 of [124]), assume that a background consisting of a non-trivial metric g and Kalb-Ramond field b is given together with Killing vectors ∂_{μ_n} such that

$$\begin{aligned} \partial_{\mu_n} g_{\alpha\beta} &= 0 \\ \partial_{\mu_n} b_{\alpha\beta} &= 0\,. \end{aligned} \tag{6.137}$$

For convenience of notation we restrict attention in the following to the coordinates x^{μ_n}, since all other coordinates are mere spectators when T-dualizing. Furthermore we will suppress the subindex n altogether.

The inner automorphism $\mathcal{T}$ of the algebra of operators on sections of the exterior bundle over loop space is defined by its action on the canonical fields as follows:

$$\begin{aligned} \mathcal{T}(-i\partial_\mu) &= X'^\mu \\ \mathcal{T}\left(X'^\mu\right) &= -i\partial_\mu \\ \\ \mathcal{T}\left(\mathcal{E}^{\dagger a}\right) &= \mathcal{E}_a \\ \mathcal{T}(\mathcal{E}_a) &= \mathcal{E}^{\dagger a}\,. \end{aligned} \tag{6.138}$$

It is possible (see [124] and pp. 47 of [8]) to express this automorphism manifestly as a similarity transformation

$$\mathcal{T}(A) = e^{-\mathbf{W}}\, A\, e^{\mathbf{W}}\,. \tag{6.139}$$

This however requires taking into account normal ordering, which would lead us too far afield in the present discussion. For our purposes it is fully sufficient to note that $\mathcal{T}$ preserves the canonical brackets

$$\begin{aligned} \left[-i\partial_{(\mu,\sigma)}, X'^{(\nu,\sigma')}\right] &= i\delta_\mu^\nu \delta'(\sigma,\sigma') \\ &= \left[\mathcal{T}\left(-i\partial_{(i,\sigma)}\right), \mathcal{T}\left(X'^{(j,\sigma')}\right)\right] \end{aligned} \tag{6.140}$$

and

$$\begin{aligned} \left\{\mathcal{E}_{(i,\sigma)}, \mathcal{E}^{\dagger(j,\sigma)}\right\} &= \delta_i^j \delta(\sigma,\sigma') \\ &= \left\{\mathcal{T}\left(\mathcal{E}_{(i,\sigma)}\right), \mathcal{T}\left(\mathcal{E}^{\dagger(j,\sigma)}\right)\right\} \end{aligned} \tag{6.141}$$

(with the other transformed brackets vanishing) and must therefore be an algebra automorphism.

Acting on the K-deformed exterior (co)derivative on loop space the transformation $\mathcal{T}$ produces (we suppress the variable σ and the mode functions ξ for convenience)

$$\begin{aligned}
\mathcal{T}(\mathbf{d}_K) &= \mathcal{T}\Big(\mathcal{E}^{\dagger a} E_a{}^\mu \partial_\mu + iT\mathcal{E}_a E^a{}_\mu X'^\mu\Big) \\
&= i\mathcal{E}_a E_a{}^\mu X'^\mu + T\mathcal{E}^{\dagger a} E^a{}_\mu \partial_\mu \\
&= \mathcal{E}^{\dagger a} \tilde{E}_a{}^\mu \partial_\mu + iT\mathcal{E}_a \tilde{E}^a{}_\mu X'^\mu
\end{aligned}$$

$$\mathcal{T}\Big(\mathbf{d}^\dagger{}_K\Big) = (\mathcal{T}(\mathbf{d}_K))^\dagger \,, \tag{6.142}$$

where the T-dual vielbein $\tilde{E}$ is defined as

$$\tilde{E}^a{}_\mu \equiv \frac{1}{T} E_a{}^\mu \,. \tag{6.143}$$

(This is obviously not a tensor equation but true in the special coordinates that have been chosen.) Therefore T-duality sends the deformed exterior derivative associated with the metric defined by the vielbein $E_a{}^\mu$ to that associated with the metric defined by the vielbein $\tilde{E}_a{}^\mu$. This yields the usual inversion of the spacetime radius $R \mapsto \alpha'/R$:

$$E^a{}_\mu \;=\; \delta^a_\mu \sqrt{2\pi} R \Rightarrow \tilde{E}^a{}_\mu \;=\; \delta^a_\mu \frac{1}{T}\frac{1}{\sqrt{2\pi}R} \;=\; \delta^a_\mu \sqrt{2\pi}\,\frac{\alpha'}{R} \,. \tag{6.144}$$

Furthermore it is readily checked that the bosonic and fermionic worldsheet oscillators transform as expected:

$$\begin{aligned}
\mathcal{T}(\mathcal{P}_{\pm,a}) &= \mathcal{T}\left(\frac{1}{\sqrt{2T}}\left(-iE_a{}^\mu \partial_\mu \pm T E_{a\mu} X'^\mu\right)\right) \\
&= \frac{1}{\sqrt{2T}}\left(E_a{}^\mu X'^\mu \pm -iT E_{a\mu}\partial_\mu\right) \\
&= \pm\frac{1}{\sqrt{2T}}\left(-i\tilde{E}_a{}^\mu \partial_\mu \pm T E_{a\mu} X'^\mu\right) \\
&= \pm\tilde{\mathcal{P}}_{\pm,a}
\end{aligned} \tag{6.145}$$

and

$$\mathcal{T}\left(\Gamma^a_\pm\right) = \pm\Gamma^a_\pm \,. \tag{6.146}$$

More generally, when the Kalb-Ramond field is included one finds

$$\begin{aligned}
\mathcal{T}\Big(\mathbf{d}_K^{(B)} \pm \mathbf{d}^{\dagger}{}_K^{(B)}\Big) &= \mathcal{T}\big(\Gamma^a_\mp E_a{}^\mu \left(\partial_\mu \mp iT\left(G_{\mu\nu} \pm B_{\mu\nu}\right) X'^\nu\right)\big) \\
&= \mp\Gamma^a_\mp E_a{}^\mu \left(iX'^\mu \mp T\left(G_{\mu\nu} \pm B_{\mu\nu}\right)\partial_\nu\right) \\
&= \Gamma^a_\mp \tilde{E}_a{}^\mu \left(\partial_\mu \mp \left[T\left(G_{\mu\nu} \pm B_{\mu\nu}\right)\right]^{-1} X'^\nu\right)
\end{aligned} \tag{6.147}$$

with

$$\tilde{E}_a{}^\mu \equiv T E_a{}^\nu (G_{\nu\mu} \pm B_{\nu\mu}) \,, \tag{6.148}$$

which reproduces the well known result (equation (2.4.39) of [135]) that the T-dual space-time metric is given by

$$\tilde{G}^{\mu\nu} = T^2[(G \mp B)G^{-1}(G \pm B)]_{\mu\nu} \tag{6.149}$$

and that the T-dual Kalb-Ramond field is

$$\begin{aligned}\tilde{B}_{\mu\nu} &= \pm[\frac{1}{T^2}(G \pm B)^{-1} - \tilde{G}]_{\mu\nu} \\ &= \left[T^2(G \mp B)B^{-1}(G \pm B)\right]^{-1}_{\mu\nu} .\end{aligned} \tag{6.150}$$

$$\tag{6.151}$$

It is also very easy in our framework to derive the Buscher rules for T-duality along a single direction y ("factorized duality"): Let $\mathcal{T}_y$ be the transformation (6.138) restricted to the ∂_y direction, then from

$$\begin{aligned}\mathcal{T}\Big(\mathbf{d}_K^{(B)} \pm \mathbf{d}^{\dagger}{}_K^{(B)}\Big) &= \Gamma^a_{\mp}\left(E_a{}^i\partial_i \mp iTE_a{}^\mu\left(G_{\mu i} \pm B_{\mu i}\right)X'^i\right) \\ &\quad + \mathcal{T}\left(\Gamma^a_{\mp}\left(E_a{}^y\partial_y \mp iTE^\mu\left(G_{\mu y} \pm B_{\mu y}\right)X'^y\right)\right) \\ &= \Gamma^a_{\mp}\left(E_a{}^i\partial_i \mp iTE_a{}^\mu\left(G_{\mu i} \pm B_{\mu i}\right)X'^i\right) \\ &\quad + \Gamma^a_{\mp}\Big(TE_a{}^\mu\left(G_{\mu y} \pm B_{\mu y}\right)\partial_y \mp iE_a{}^y e^{\Phi/2}X'^y\Big)\end{aligned} \tag{6.152}$$

one reads off the T-dual inverse vielbein

$$\begin{aligned}\tilde{E}_a{}^i &= E_a{}^i \\ \tilde{E}_a{}^y &= TE_a{}^\mu\left(G_{\mu y} \pm B_{\mu y}\right)\end{aligned} \tag{6.153}$$

whose inverse $\tilde{E}_\mu{}^a$ is easily seen to be

$$\begin{aligned}\tilde{E}_i{}^a &= E_i{}^a - \frac{G_{iy} \pm B_{iy}}{G_{yy}}E_y{}^a \\ \tilde{E}_y{}^a &= \frac{1}{TG_{yy}}E_y{}^a ,\end{aligned} \tag{6.154}$$

which gives the T-dual metric with minimal computational effort:

$$\begin{aligned}\tilde{G}_{yy} &= \frac{1}{TG_{yy}} \\ \tilde{G}_{iy} &= \mp\frac{B_{iy}}{TG_{yy}} \\ \tilde{G}_{ij} &= G_{ij} - \frac{1}{G_{yy}}\left(G_{iy}G_{jy} - B_{iy}B_j y\right) .\end{aligned} \tag{6.155}$$

Similarly the relations

$$\begin{aligned}\tilde{E}_a{}^\mu(\tilde{G}_{\mu i} \pm \tilde{B}_{\mu i}) &= E_a{}^\mu(G_{\mu i} \pm B_{\mu i}) \\ \tilde{E}_a{}^\mu(\tilde{G}_{\mu y} \pm \tilde{B}_{\mu y}) &= \frac{1}{T}E_a{}^y\end{aligned} \tag{6.156}$$

for the T-dual B-field $\tilde{B}$ are read off from (6.152). Solving them for $\tilde{B}$ is straightforward and yields

$$\begin{aligned}\tilde{B}_{ij} &= B_{ij} \mp \frac{1}{G_{yy}} \left(B_{jy}G_{iy} - B_{iy}G_{iy}\right) \\ \tilde{B}_{iy} &= \frac{1}{TG_{yy}} G_{iy}\,.\end{aligned} \tag{6.157}$$

These are the well known *Buscher rules* for factorized T-duality (see eq. (4.1.9) of [135]).

The constant dilaton can be formally absorbed into the string tension T and is hence seen to be invariant under $\mathcal{T}_y$. This is correct in the classical limit that we are working in. It is well known (e.g. eq. (4.1.10) of [135]), that there are higher loop corrections to the T-dual dilaton. These corrections are not visible with the methods discussed here.

Using our representation for the superconformal generators in various backgrounds it is now straightfroward to include more general background fields than just G and B in the above construction:

6.4.2.2 T-duality for various backgrounds. When turning on all fields G, B, A, C and Φ, requiring them to be constant in the sense of (6.137) and assuming for convenience of notation that $B{\cdot}C = 0$ the supercurrents read according to the considerations in §6.3.1-§6.3.3.4

$$\mathbf{d}_K^{(\Phi)(A+B+C)} \pm \mathbf{d}^{\dagger}{}_K^{(\Phi)(A+B+C)} = \Gamma^a_{\mp} E_a{}^{\mu} \left(e^{\Phi/2}(G_\mu{}^\nu \pm C_\mu{}^\nu)\partial_\nu \mp iTe^{-\Phi/2}(G_{\mu\nu} \pm (B_{\mu\nu} + \frac{1}{T}F_{\mu\nu})X'^\nu \right)\,. \tag{6.158}$$

It is straightforward to apply $\mathcal{T}$ to this expression and read off the new fields. However, since the resulting expressions are not too enlightening we instead use a modification $\tilde{\mathcal{T}}$ of $\mathcal{T}$, which, too, induces an algebra isomorphism, but which produces more accessible field redefinitions. The operation $\tilde{\mathcal{T}}$ differs from $\mathcal{T}$ in that index shifts are included:

$$\begin{aligned}\tilde{\mathcal{T}}(-i\partial_\mu) &\equiv T g_{\mu\nu} X'^\nu \\ \tilde{\mathcal{T}}\big(X'^\mu\big) &\equiv -\frac{i}{T} g^{\mu\nu}\partial_\nu \\ \\ \tilde{\mathcal{T}}\Big(\mathcal{E}^{\dagger a}\Big) &\equiv \mathcal{E}^a \\ \tilde{\mathcal{T}}(\mathcal{E}^a) &\equiv \mathcal{E}^{\dagger a}\,.\end{aligned} \tag{6.159}$$

Due to the constancy of $g_{\mu\nu}$ this preserves the canonical brackets just as in (6.140) and hence is indeed an algebra isomorphism.

Applying it to the supercurrents (6.158) yields

$$\tilde{\mathcal{T}}\left[\mathbf{d}_K^{(\Phi)(B+C)} \pm \mathbf{d}_K^{(\Phi)(B+C)}\right] = \Gamma^a_{\mp} E_a{}^{\mu} \left(e^{-\Phi/2}(G_\mu{}^\nu \pm (B_\mu{}^\nu + \frac{1}{T}F_\mu{}^\nu)\partial_\nu \mp iTe^{\Phi/2}(G_{\mu\nu} \pm C_{\mu\nu})X'^\nu \right)\,. \tag{6.160}$$

Comparison shows that under $\tilde{\mathcal{T}}$ the background fields transform as

$$
\begin{aligned}
B_{\mu\nu} + \frac{1}{T}F_{\mu\nu} &\to C_{\mu\nu} \\
C_{\mu\nu} &\to B_{\mu\nu} + \frac{1}{T}F_{\mu\nu} \\
G_{\mu\nu} &\to G_{\mu\nu} \\
\Phi &\to -\Phi\,.
\end{aligned}
\tag{6.161}
$$

The fact that under this transformation the NS-NS 2-form is exchanged with what we interpreted as the R-R 2-form and that the dilaton reverses its sign is reminiscent of S-duality. It is well known [136] that T-duality and S-duality are themselves dual under the exchange of the fundamental F-string and the D-string. How exactly this applies to the constructions presented here remains to be investigated. (For instance the sign that distinguishes (6.161) from the expected result would need to be explained, maybe by a change of orientation of the string.)

6.4.3 Hodge Duality on Loop Space

For the sake of completeness in the following the relation of loop space Hodge duality to the above discussion is briefly indicated. It is found that ordinary Hodge duality is at least superficially related to the algebra isomorphisms discussed in §6.4.2 (p.121). Furthermore a deformed version of Hodge duality is considered which preserves the familiar relation $\mathbf{d}^\dagger = \pm \star\, \mathbf{d} \star^{-1}$.

6.4.3.1 Ordinary Hodge duality. On a finite dimensional pseudo-Riemannian manifold, let $\bar{\star}$ be the phase-shifted Hodge star operator which is normalized so as to satisfy

$$
\begin{aligned}
(\bar{\star})^\dagger &= -\bar{\star} \\
(\bar{\star})^2 &= 1\,.
\end{aligned}
\tag{6.162}
$$

(For the precise relation of $\bar{\star}$ to the ordinary Hodge $\star$ see (A.18) of [113].) The crucial property of this operator can be expressed as

$$
\bar{\star}\, \hat{\mathrm{e}}^{\dagger\mu} = \hat{\mathrm{e}}^{\mu}\, \bar{\star}\,,
\tag{6.163}
$$

where $\hat{\mathrm{e}}^{\dagger\mu}$ is the operator of exterior multiplication by dx^μ and $\hat{\mathrm{e}}^\mu$ is its adjoint under the Hodge inner product.

It has been pointed out in [3] that the notion of Hodge duality can be carried over to infinite dimensional manifolds. This means in particular that on loop space there is an idempotent operator $\bar{\star}$ so that

$$
\bar{\star}\, \mathcal{E}^{\dagger\mu} = \mathcal{E}^{\mu}\, \bar{\star}
\tag{6.164}
$$

and

$$
\left[\bar{\star}, X^{(\mu,\sigma)}\right] = 0 = \left[\bar{\star}, \hat{\nabla}_{(\mu,\sigma)}\right]\,.
\tag{6.165}
$$

It follows in particular that the K-deformed exterior derivative is related to its adjoint by

$$\mathbf{d}^{\dagger}{}_K = -\bar{\star}\, \mathbf{d}_K\, \bar{\star}\,. \tag{6.166}$$

In fact this holds for all the modes:

$$\mathbf{d}^{\dagger}{}^{\dagger}_{K,\xi} = -\bar{\star}\, \mathbf{d}_{K,\xi}\, \bar{\star}\,. \tag{6.167}$$

In the spirit of the discussion of T-duality by algebra isomorphisms in §6.4.2 (p.121) one can equivalently say that $\bar{\star}$ induces an algebra isomorphism $\mathcal{H}$ defined by

$$\mathcal{H}(A) \equiv \bar{\star}\, A \bar{\star}\,, \tag{6.168}$$

i.e.

$$\begin{aligned} \mathcal{H}(-i\partial_\mu) &= -i\partial_\mu \\ \mathcal{H}(X^\mu) &= X^\mu \\ \\ \mathcal{H}\left(\mathcal{E}^{\dagger a}\right) &= \mathcal{E}^a \\ \mathcal{H}(\mathcal{E}^a) &= \mathcal{E}^{\dagger a}\,. \end{aligned} \tag{6.169}$$

It is somewhat interesting to consider the result of first applying $\mathcal{H}$ to $\mathbf{d}_K$ and then acting with the deformation operators $\exp(\mathbf{W})$ considered before. This is equivalent to considering the deformation obtained by $\bar{\star}\, e^{\mathbf{W}}$. This yields

$$\begin{aligned} \mathbf{d}_K &\to \left(e^{-\mathbf{W}}\bar{\star}\right) \mathbf{d}_K \left(\bar{\star}\, e^{\mathbf{W}}\right) \;=\; -e^{-\mathbf{W}}\, \mathbf{d}^{\dagger}{}_K\, e^{\mathbf{W}} \\ \mathbf{d}^{\dagger}{}_K &\to \left(e^{\mathbf{W}^\dagger}\bar{\star}\right) \mathbf{d}^{\dagger}{}_K \left(\bar{\star}\, e^{-\mathbf{W}^\dagger}\right) \;=\; -e^{\mathbf{W}^\dagger}\, \mathbf{d}_K\, e^{-\mathbf{W}^\dagger}\,. \end{aligned} \tag{6.170}$$

Hence, except for a global and irrelevant sign, the deformations induced by $e^{\mathbf{W}}$ and $\bar{\star}\, e^{\mathbf{W}}$ are related by

$$\mathbf{W} \leftrightarrow -\mathbf{W}^\dagger\,. \tag{6.171}$$

Looking back at the above results for the backgrounds induced by various $\mathbf{W}$ this corresponds to

$$\begin{aligned} B &\leftrightarrow C \\ A &\leftrightarrow A \\ \Phi &\leftrightarrow -\Phi\,. \end{aligned} \tag{6.172}$$

It should be noted though, that unlike the similar correspondence (6.161) both sides of this relation are not unitarily equivalent in the sense that the corresponding superconformal generators $e^{-\mathbf{W}}\bar{\star}\, \mathbf{d}_K\, \bar{\star} e^{\mathbf{W}}$ and $e^{-\mathbf{W}}\, \mathbf{d}_K\, e^{\mathbf{W}}$ are not unitarily equivalent.

Nevertheless, it might be that the physics described by both generators is somehow related. This remains to be investigated.

6.4.3.2 Deformed Hodge duality. The above shows that for general background fields (general deformations of the superconformal algebra) the familiar equality of ${\mathbf{d}^\dagger}^{\mathbf{W}}_{K,\xi}$ with $-\bar{\star}\,\mathbf{d}^{\mathbf{W}}_{K,\xi}\,\bar{\star}^{-1}$ is violated. It is however possible to consider a deformation $\bar{\star}^{\mathbf{W}}$ of $\bar{\star}$ itself which restores this relation:

$$\bar{\star}^{\mathbf{W}} \equiv e^{\mathbf{W}^\dagger}\,\bar{\star}\,e^{\mathbf{W}}\,. \tag{6.173}$$

Obviously this operator satisfies

$${\mathbf{d}^\dagger}^{\mathbf{W}}_{K,\xi} = -\bar{\star}^{\mathbf{W}}\,\mathbf{d}_{K,\xi}\,(\bar{\star}^{\mathbf{W}})^{-1}\,. \tag{6.174}$$

The Hodge star remains invariant under this deformation when $\mathbf{W}$ is anti-Hodge-dual:

$$\bar{\star} = \bar{\star}^{\mathbf{W}} \Leftrightarrow \bar{\star}\,\mathbf{W}\,\bar{\star} = -\mathbf{W}^\dagger\,. \tag{6.175}$$

This is in particular true for the gravitational deformation of §6.3.3.1 (p.110). It follows that $\bar{\star}^{(G)} = \bar{\star}$. This can be understood in terms of the fact that the definition of the Hodge star involves only the orthonormal metric on the tangent space (*cf.* (A.14) of [113]).

6.4.4 Deformed inner Products on Loop Space.

The above discussion of deformed Hodge duality on loop space motivates the following possibly interesting observation:

From the point of view of differential geometry the exterior derivative $\mathbf{d}$ on a manifold is a purely topological object which does not depend in any way on the geometry, i.e. on the metric tensor. The geometric information is instead contained in the Hodge star operator $\star$, the Hodge inner product $\langle\alpha|\beta\rangle = \int_M \alpha\wedge\star\beta$ on differential forms and the adjoint $\mathbf{d}^\dagger$ of $\mathbf{d}$ with respect to $\langle\cdot|\cdot\rangle$.

We have seen in §6.4.3.2 (p.128) that deformations of the Hodge star operator on loop space may encode not only information about the geometry of target space, but also about other background fields, like Kalb-Ramond and dilaton fields. These deformations are accompanied by analogous deformations (6.65) of $\mathbf{d}$ and $\mathbf{d}^\dagger$.

But from this point of view of differential geometry it appears unnatural to associate a deformation of both $\mathbf{d}^\dagger$ as well as $\mathbf{d}$ with a deformed Hodge star operator. One would rather expect that $\mathbf{d}$ remains unaffected by any background fields while the information about these is contained in $\star$, $\langle\cdot|\cdot\rangle$ and $\mathbf{d}^\dagger$.

Here we want to point out that both points of view are equivalent and indeed related by a global similarity transformation ('duality') and that the change in point of view makes an interesting relation to noncommutative geometry transparent.

Namely consider deformed operators

$$\begin{aligned}\mathbf{d}^{(\mathbf{W})} &= e^{-\mathbf{W}}\mathbf{d}e^{\mathbf{W}}\\ {\mathbf{d}^\dagger}^{(\mathbf{W})} &= e^{\mathbf{W}^\dagger}\mathbf{d}^\dagger e^{-\mathbf{W}^\dagger}\end{aligned} \tag{6.176}$$

on an inner product space $\mathcal{H}$ with inner product $\langle\cdot|\cdot\rangle$ as in (6.65).

By applying a global similarity transformation

$$\begin{aligned} |\psi\rangle &\to |\tilde{\psi}\rangle \equiv e^{\mathbf{W}}\,|\psi\rangle \\ A &\to \tilde{A} \equiv e^{\mathbf{W}} A e^{-\mathbf{W}} \end{aligned} \tag{6.177}$$

to all elements $|\psi\rangle \in \mathcal{H}$ and all operators A on $\mathcal{H}$ one of course finds

$$\begin{aligned} \left(\mathbf{d}^{(\mathbf{W})}\right)^{\tilde{}} &= \mathbf{d} \\ \left(\mathbf{d}^{\dagger(\mathbf{W})}\right)^{\tilde{}} &= e^{\mathbf{W}+\mathbf{W}^\dagger}\mathbf{d}^\dagger e^{-\mathbf{W}-\mathbf{W}^\dagger} \,. \end{aligned} \tag{6.178}$$

By construction, the algebra of $\left(\mathbf{d}^{(\mathbf{W})}\right)^{\tilde{}}$ and $\left(\mathbf{d}^{\dagger(\mathbf{W})}\right)^{\tilde{}}$ is the same as that of $\mathbf{d}^{(\mathbf{W})}$ and $\mathbf{d}^{\dagger(\mathbf{W})}$. But now all the information about the deformation induced by $\mathbf{W}$ is contained in $\left(\mathbf{d}^{\dagger(\mathbf{W})}\right)^{\tilde{}}$ alone. This has the advantage that we can consider a deformed innner product

$$\langle\cdot|\cdot\rangle_{(\mathbf{W})} \equiv \langle\cdot|\, e^{-(\mathbf{W}+\mathbf{W}^\dagger)}\,\cdot\rangle \tag{6.179}$$

on $\mathcal{H}$ with respect to which

$$\mathbf{d}^{\dagger_{(\mathbf{W})}} = \left(\mathbf{d}^{\dagger(\mathbf{W})}\right)^{\tilde{}} \,, \tag{6.180}$$

where $\langle A\cdot|\cdot\rangle_{(\mathbf{W})} \equiv \langle\cdot|A^{\dagger_{(\mathbf{W})}}|\cdot\rangle_{(\mathbf{W})}$. If the original inner product came from a Hodge star this corresponds to a deformation

$$\star \to \star\, e^{-(\mathbf{W}+\mathbf{W}^\dagger)} \,. \tag{6.181}$$

This way now indeed the entire deformation comes from a deformation of the Hodge star and the inner product.

That this is equivalent to the original notion (6.176) of deformation can be checked again by noting that the deformed inner product of the deformed states agrees with the original inner prodcut on the original states

$$\langle\tilde{\psi}|\tilde{\phi}\rangle_{(\mathbf{W})} = \langle\psi|\phi\rangle \,. \tag{6.182}$$

These algebraic manipulations by themselves are rather trivial, but the interesting aspect is that the form (6.179) of the deformation appears in the context of noncommutative spectral geometry [39]. The picture that emerges is roughly that of a spectral triple $(\mathcal{A}, \mathbf{d}_K \pm \mathbf{d}_K^{\dagger_{(\mathbf{W})}}, \mathcal{H})$, where $\mathcal{A}$ is an algebra of functions on loop space (*cf.* 6.2.1 (p.98)), $\mathcal{H}$ is the inner product space of differential forms over loop space equipped with a deformed Hodge inner product (6.179) which encodes all the information of the background fields on target space, and where two Dirac operators are given by $\mathbf{d}_K \pm \mathbf{d}_K^{\dagger_{(\mathbf{W})}}$. There have once been attempts [10, 11, 6, 5, 123] to understand the superstring by regarding the worldsheet supercharges as Dirac operators in a spectral triple. Maybe the insight that and how target space background fields manifest themselves as simple algebraic deformations (6.65) of the Dirac operators, or, equivalently, (6.179) of the inner product on $\mathcal{H}$ can help to make progress with this approach.

6.5 Appendix: Canonical Analysis of Bosonic D1 Brane Action

The bosonic part of the worldsheet action of the D-string is

$$\mathcal{S} = -T\int d^2\sigma\, e^{-\Phi}\sqrt{-\det(G_{ab}+B_{ab}+\frac{1}{T}F_{ab})} + T\int\Big(C_2 + C_0(B+\frac{1}{T}F)\Big) \quad (6.183)$$

where G, B, C_0 and C_2 are the respective background fields and $F_{ab} = (dA)_{ab}$ is the gauge field on the worldsheet. Indices a, b range over the worldsheet dimensions and indices μ, ν over target space dimensions.

Using *Nambu-Brackets* $\{X^\mu, X^\nu\} \equiv \epsilon^{ab}\partial_a X^\mu\, \partial_b X^\nu$ (with $\epsilon^{01} = 1$, $\epsilon^{ab} = -\epsilon^{ba}$) the term in the square root can be rewritten as

$$-\det\Big(G_{ab}+B_{ab}+\frac{1}{T}F_{ab}\Big) = -\frac{1}{2}\{X^\mu, X^\nu\}G_{\mu\mu'}G_{\nu\nu'}\{X^{\mu'}, X^{\nu'}\} - (B_{01}+\frac{1}{T}F_{01})^2 \quad (6.184)$$

The canonical momenta associated with the embedding coordinates X^μ are

$$\begin{aligned} P_\mu &= \frac{\delta\mathcal{L}}{\delta\dot{X}^\mu} \\ &= T\left(\frac{1}{e^\Phi\sqrt{-\det(G+B+F/T)}}\Big(X'^\nu G_{\mu\mu'}G_{\nu\nu'}\{X^{\mu'}, X^{\nu'}\} + B_{\mu\nu}X'^\nu(B_{01}+\frac{1}{T}F_{01})\Big)\right) + \\ &\quad +T(C_2 + C_0 B)_{\mu\nu}X'^\nu\,. \end{aligned} \quad (6.185)$$

On the other hand the canonical momenta associated with the gauge field read

$$\begin{aligned} E_0 &\equiv \frac{\delta\mathcal{L}}{\delta\dot{A}_1} = 0 \\ E_1 &\equiv \frac{\delta\mathcal{L}}{\delta\dot{A}_1} = \frac{1}{e^\Phi\sqrt{-\det(G+B+F/T)}}(B_{01}+\frac{1}{T}F_{01}) + C_0\,. \end{aligned} \quad (6.186)$$

Since the gauge group is U(1), A_μ is a periodic variable and hence the eigenvalues of E_1 are discrete [137]:

$$E_1 \equiv p \in \mathbb{Z}\,. \quad (6.187)$$

Inverting (6.186) allows to rewrite the canonical momenta P_μ as

$$P_\mu = \frac{1}{\sqrt{-\det(G)}}\tilde{T}X'^\nu G_{\mu\mu'}G_{\nu\nu'}\{X^{\mu'}, X^{\nu'}\} + T(C_2 + pB)_{\mu\nu}X'^\nu\,, \quad (6.188)$$

where

$$\tilde{T} \equiv T\sqrt{e^{-2\Phi} + (p - C_0)^2} \quad (6.189)$$

is the tension of a bound state of one D-string with p F-strings [138]. In this form it is easy to check that the following two constraints are satisfied:

$$\begin{aligned} &(P - T(C_2+pB)\cdot X')\cdot(P - T(C_2+pB)\cdot X') + \tilde{T}^2 X'\cdot X' = 0 \\ &(P - T(C_2+pB)\cdot X')\cdot X' = 0\,, \end{aligned} \quad (6.190)$$

which express temporal and spatial reparameterization invariance, respectively. For constant $\tilde{T}$ this differes from the familiar constraints for the pure F-string only in a redefinition of the tension and the couplings to the background 2-forms.

For non-constant $\tilde{T}$, however, things are a little different. For the purpose of comparison with the results in §6.3.3.3 (p.112) consider the case $B = C_0 = C_2 = p = 0$ and Φ possibly non-constant. In this case the constraints (6.190) can be equivalently rewritten as

$$\mathcal{P}_{\pm}{}^2 = 0 \tag{6.191}$$

with

$$\mathcal{P}_{\mu,\pm} = e^{\Phi/2} P_\mu \pm T e^{-\Phi/2} G_{\mu\nu} X'^\nu \,. \tag{6.192}$$

Up to fermionic terms this is the form found in (6.89).

7. Worldsheet Invariants and Boundary States

The following is taken from [28], which was mainly a reaction to attempts [139, 140, 141, 142, 143] to find an alternative quantization of the string in terms of non-standard invariants. Only after [28] was published did I become aware that essentially the same result had appeared long before in [144, 145].

7.1 DDF and Pohlmeyer invariants

The classical string is known to be a completely integrable system with an infinite number of classical observables that Poisson-commute with all the constraints. A concise and comprehensive review of the work by Pohlmeyer, Rehren, Bahns, et. al. [140, 141, 142, 143] on a particular manifestation of these *gauge invariant observables*, known as *Pohlmeyer invariants*, is given in [139].

Since the Virasoro algebra is the direct sum of two copies of $\mathrm{diff}(S^1)$, the diffeomorphism algebra of the circle, invariant observables are simply those that are reparameterization invariant with respect to these two algebras. Two common types of reparameterization invariant objects are

- Wilson lines,
- integrals over densities of unit reparameterization weight .

More precisely (see §7.1.1.2 (p.137) for a detailed derivation), let $\mathcal{P}^\mu_\pm(\sigma)$ be the left- and right-moving classical fields on the free closed bosonic string with Poisson brackets of the form

$$\left[\mathcal{P}^\mu_\pm(\sigma), \mathcal{P}^\nu_\pm(\sigma')\right]_{\mathrm{PB}} = \pm\eta^{\mu\nu}\delta'(\sigma,\sigma') \ . \tag{7.1}$$

These transform with unit weight under the action of the Virasoro algebra and hence the *Wilson line*

$$\mathrm{Tr}\mathbf{P}\exp\left(\int\limits_{S^1} \mathcal{P}^\mu_+ A_{+\mu}\right)\mathrm{Tr}\mathbf{P}\exp\left(\int\limits_{S^1} \mathcal{P}^\mu_- A_{-\mu}\right) , \tag{7.2}$$

Poisson-commutes with all Virasoro constraints (where $\mathbf{P}$ denotes path-ordering and $A_{\pm\mu}$ are two *constant* Lie-algebra-valued 1-forms on target space). It is easy to see that also the coefficients of $\mathrm{Tr}(A^n)$ in the Taylor expansion of this object commute with the constraints. These coefficients are known as the *Pohlmeyer invariants*. The Poisson algebra of these observables is rather convoluted. The problem of finding a quantum deformation of this algebra turns out to be difficult and involved and has up to now remained unsolved [141, 139, 146]. Furthermore, by itself, it is not obvious how the above construction should generalize to the superstring.

For these reasons it seems worthwhile to consider the possibility of alternatively using integrals over unit weight densities to construct a complete set of classical invariant charges.

A little reflection shows that the well-known *DDF operators* [147] for the covariantly quantized string, which are operators that commute with all the quantum (super-)Virasoro constraints, are built using essentially this principle:

From elementary CFT it follows that for $\mathcal{O}(z)$ any primary CFT field of conformal weight $h = 1$ we have (after the usual introduction of a complex coordinate z on the worldsheet)

$$\left[L_n, \oint dz\, \mathcal{O}(z)\right] = 0\,, \qquad \forall\, n\,. \tag{7.3}$$

By choosing $\mathcal{O}(z) = \left[G_{-\nu}, \tilde{\mathcal{O}(z)}\right]$, (with $G_{-\nu}$ the ($\nu = 0$)-mode (R sector) or ($\nu = -1/2$)-mode (NS sector) of the worldsheet supercurrent and $h\left(\tilde{\mathcal{O}}\right) = 1/2$) this generalizes to the superstring

$$\left[G_{n-\nu}, \oint dz\, \mathcal{O}(z)\right] = 0\,, \qquad \forall\, n\,, \tag{7.4}$$

as is reviewed below in §7.1.1.1. It hence only remains to find $\mathcal{O}$ (or $\tilde{\mathcal{O}}$) of weight 1 (or 1/2) such that the resulting integrals have nice (super-)commutators and exhaust the space of all invariant charges. Doing this in a natural way yields the DDF operators.

It is readily checked that this construction of the DDF operators can be mimicked in terms of the classical Poisson algebra to yield a complete set of classical invariants which we shall call *classical DDF invariants*. These inherit all the nice properties of their quantum cousins.

Most importantly, as is shown in §7.1.2 (p.140), the Pohlmeyer invariants can be expressed in terms of the classical DDF invariants. Since it is known how the latter have to be quantized (i.e. the crucial quantum corrections to these charges is well known *cf.* §7.1.1.1 (p.134)) this also tells us how the Pohlmeyer invariants can consistently be quantized.

In particular this shows that any normal ordering in the quantization of the Pohlmeyer invariants must be applied only inside each DDF operator, while the DDF operators among themselves need not be reordered. This clarifies the result of [146], where it was demonstrated that the Pohlmeyer invariants cannot be consistently quantized by writing them in terms of worldsheet oscillators and applying normal ordering with respect to these. Rather, as will be shown here, one has to replace these oscillators with the corresponding DDF observables, and the assertion is that the Pohlmeyer invariants, like any other reparameterization invariant observable, are unaffected by this replacement.

Because the DDF operators, together with the identity operator, form a closed algebra, the quantization of the Pohlmeyer invariants in terms of DDF operators, as demonstrated here, is manifestly consistent in the sense that the quantum commutator of two such invariants is itself again an invariant.

It should be emphasized that, in contrast to what has been stated in [146], the construction of classical DDF invariants does *not* require that any worldsheet coordinate gauge has to be fixed, in particular their construction has nothing to do with fixing conformal gauge. This is obvious due to the fact that the classical DDF invariants are constructable

(as the Pohlmeyer invariants, too) by proceeding from just the Nambu-Goto action, which does not even have an auxiliary worldsheet metric which could be gauge fixed. Furthermore the canonical data and the form of the Virasoro constraints as obtained from the Nambu-Goto action are precisely the same as those obtained from the Polyakov action with or without fixed worldsheet gauge.

Furthermore, our proof that the Pohlmeyer invariants can be equivalently expressed in terms of DDF invariants (i.e. certain polynomials in DDF invariants are equal to the Pohlmeyer invariants) constructively demonstrates that both are on the same footing as far as requirements for their respective construction is concerned.

The organization of this paper is as follows:

We first review the construction of DDF operators in §7.1.1.1 (p.134) and then that of Pohlmeyer invariants in §7.1.1.2 (p.137).

Then in §7.1.2.1 (p.140) we discuss the classical DDF invariants in detail, show how they can be used to express the Pohlmeyer invariants (§7.1.2.2 (p.144)) and how this generalizes to the superstring (§7.1.2.3 (p.145)).

A brief summary of the results presented here will be published in [29].

7.1.1 DDF operators and Pohlmeyer invariants

We review first the DDF operators, then the Pohlmeyer invariants, then show how both are related.

7.1.1.1 DDF operators. The construction of DDF operators [147] is very well known, but to the best of our knowledge there is no comprehensive review of all possible cases (transversal and longitudinal, bosonic and fermionic) available in the standard literature. The following section tries to list and derive all the essential facts.

In the standard textbook literature one can find

- in [148] (in non-CFT language) the construction of
  - transversal bosonic (§2.3.2)
  - transversal supersymmetric (§4.3.2)
  - longitudinal bosonic (pp. 111),
- and in [129] (in CFT language) the construction of
  - transversal bosonic (eq. (8.2.29))

  DDF states, which go back to [147].

The following summarizes and derives (in CFT language) *all*

- transversal and longitudinal
- bosonic and fermionic

DDF operators (for a free supersymmetric worldsheet theory).

Using the standard normalization of the OPE

$$\begin{aligned} X^\mu(z)\, X^\nu(0) &\sim -\frac{\alpha'}{2}\eta^{\mu\nu}\ln z \\ \psi^\mu(z)\,\psi^\nu(0) &\sim \frac{\eta^{\mu\nu}}{z} \end{aligned} \tag{7.5}$$

for the bosonic and fermionic worldsheet fields, the (super-)Virasoro currents read

$$\begin{aligned} T(z) &= -\frac{1}{\alpha'}\partial X\cdot\partial X(z) - \frac{1}{2}\psi\cdot\partial\psi \\ T_{\rm F}(z) &= i\sqrt{\frac{\alpha'}{2}}\psi\cdot\partial X\,. \end{aligned} \tag{7.6}$$

The *DDF operators* are defined as a set of operators that commute with all modes of T and $T_{\rm F}$ (are gauge invariant observables) and satisfy an algebra that mimics that of worldsheet oscillator creation/annihilation operators.

First of all one needs to single out two linearly independent lightlike Killing vectors l and k on target space, and in the context of this subsection we choose to normalize them as $l{\cdot}k = 2$. The span of l and k is called the *longitudinal* space and its orthogonal complement is the *tranverse* space.

For $\mathcal{O}(z) = \sum\limits_{-\infty}^{\infty} \mathcal{O}_m z^{-(m+h)}$ a primary field of weight h we shall refer to the OPE $T(z)\,\mathcal{O}(0) \sim \frac{h}{z^2}\mathcal{O}(0) + \frac{1}{z}\partial\mathcal{O}(0)$ as the *tensor law* in some of the following formulas, instead of writing out all terms. The modes of T and $T_{\rm F}$ are denoted by L_m and $G_{m-\nu}$ as usual.

The elementary but crucial fact used for the construction of DDF operators is that 0-modes of tensor operators of weight $h = 1$ commute with all L_m generators according to

$$[L_m, \mathcal{O}_n] = ((h-1)m - n)\,\mathcal{O}_{m+n}\,. \tag{7.7}$$

Therefore the task of finding DDF states is reduced to that of finding linearly independent $h = 1$ fields that have the desired commutation relations and, in the case of the superstring, are closed with respect to $T_{\rm F}$ (see below).

7.1.1.1.1 Bosonic string. For the bosonic string the DDF operators A_n^μ are defined by

$$A_n^\mu \propto \oint \frac{dz}{2\pi i}\left(\partial X^\mu + k^\mu\frac{\alpha'}{8}in\partial\ln\left(k\cdot\partial X\right)\right) e^{inkX}(z)\,. \tag{7.8}$$

(These are of course nothing but integrated vertex operators of the massless fields. Note that the logarithmic terms of $k{\cdot}\partial X$, as well as the inverse powers that will be used further below, are well defined operators, as is discussed above equation (2.3.87) in [148].)

It is straightforward to check that the operators (7.8) are really invariant:

First consider the transverse DDF operators. For v a transverse target space vector (such that in particular $v\cdot k = 0$) the operator $v\cdot A_n$ is manifestly the 0-mode of an $h = 1$ primary field (the exponential factor has $h = 0$ due to $k\cdot k = 0$) and hence is invariant.

Furthermore $k\cdot A_n \propto \delta_{n,0} k\cdot\oint \partial X$ (for $n\neq 0$ the integrand is a total derivative) also obvioulsy commutes with the L_m.

The only subtlety arises for the longitudinal $l\cdot A_n$. Here, the non-tensor behaviour of

$$T(z)\, l\cdot\partial X e^{inkX}(w) \sim -\frac{\alpha'}{2}\frac{in}{(z-w)^3}e^{inkX}(w) + (h=1)\text{-tensor law} \tag{7.9}$$

is precisely canceled by the curious logarithmic correction term $\partial \ln(k\cdot\partial X)(z) = \frac{k\partial^2 X}{k\partial X}(z)$. Namely because of

$$T(z)\,\partial^2 X^\mu(w) \sim \frac{2\partial X^\mu(w)}{(z-w)^3} + (h=2)\text{-tensor law} \tag{7.10}$$

one has

$$\Rightarrow T(z)\,\frac{k\cdot\partial^2 X}{k\cdot\partial X}e^{inkX}(w) \sim \frac{2e^{inkX}}{(z-w)^3} + (h=1)\text{-tensor law}\,, \tag{7.11}$$

which hence makes the entire integrand of $l\cdot A_m$ transform as an $h=1$ primary, as desired.

7.1.1.1.2 Superstring The analogous construction for the superstring has to ensure in addition that the DDF operators commute with the supercharges $G_{m-\nu}$. This is simply achieved by 'closing' the integral over a given weight $h=1/2$ primary field $D(z)$ to obtain the operator

$$[G_{-\nu}, D_\nu]_\iota = \left[\oint \frac{dz}{2\pi i}T_F(z), \oint \frac{dz}{2\pi i}D(z)\right] \qquad \begin{cases}\nu=0 & \text{R sector}\\ \nu=1/2 & \text{NS sector}\end{cases}\,. \tag{7.12}$$

Here and in the remainder of this subsection the brackets denote supercommutators.

The resulting operator is manifestly the zero mode of a weight $h=1$ tensor and hence commutes with all L_n. Furthermore it commutes with $G_{-\nu}$ because of

$$[G_{-\nu}, [G_{-\nu}, D_\nu]_\iota]_\iota = [L_{-2\nu}, D_\nu]_\iota \overset{(7.7)}{=} 0\,. \tag{7.13}$$

Since $G_{-\nu}$ and $L_m, \forall\, m$ generate the entire algebra, the 'closed' operator $[G_{-\nu}, D_\nu]$ indeed commutes with all L_m and $G_{m-\nu}\,, \forall\, m$.

It is therefore clear that the superstring DDF operators, which can be defined as

$$\begin{aligned} A_n^\mu &\equiv \left[G_\nu, \oint \frac{dz}{2\pi i}\psi^\mu e^{inkX}(z)\right] \\ B_n^\mu &\equiv \left[G_\nu, \oint \frac{dz}{2\pi i}\left(\psi^\mu\, k\cdot\psi - \frac{1}{4}k^\mu \partial \ln(k\cdot\partial X)\right)\frac{e^{inkX}}{\sqrt{k\cdot\partial X}}\right] \end{aligned} \tag{7.14}$$

commute with the super-Virasoro generators, since the second arguments of the commutators are integrals over weight 1/2 tensors. (And of course the latter are nothing but the integrated vertex operators as they appear in the (-1) superghost picture). The nature and purpose of the logarithmic correction term in the second line is just as discussed for the bosonic theory above: It cancels the non-tensor term in

$$T(z)\; l\cdot\psi\, k\cdot\psi \frac{e^{ikX}}{\sqrt{k\cdot\partial X}}(w) \sim \frac{1}{(z-w)^3}\frac{e^{inkX}}{\sqrt{k\cdot\partial X}} + (h=1/2)\text{-tensor law}\,. \tag{7.15}$$

Evaluating the above supercommutators yields the explicit form for A_n^μ and B_n^μ:

$$
\begin{aligned}
A_n^\mu &= i\sqrt{\frac{2}{\alpha'}}\oint \frac{dz}{2\pi i}\left(\partial X^\mu + \frac{\alpha'}{2} in\psi^\mu\, k\cdot\psi\right) e^{inkX}(z)\\
B_n^\mu &= i\sqrt{\frac{2}{\alpha'}}\oint \frac{dz}{2\pi i}\left(\partial X^\mu\, k\cdot\psi - \psi^\mu k\cdot\partial X + \frac{\alpha'}{4}\psi^\mu\, k\cdot\psi\, k\cdot\partial\psi \frac{1}{k\cdot\partial X}\right)\frac{e^{inkX}}{\sqrt{k\cdot\partial X}}(z)\\
&\quad + i\sqrt{\frac{2}{\alpha'}}\oint \frac{dz}{2\pi i} k^\mu \left(k\cdot\psi f_1(k\cdot X, k\cdot\partial X) + k\cdot\partial\psi f_2(k\cdot X, k\cdot\partial X)\right)(z)\;,
\end{aligned}
\tag{7.16}
$$

where f_1 and f_2 are functions which we don not need to write out here.

The above discussion has focused on only a single chirality sector (left-moving, say). It must be noted that the exponent $ink\cdot X$ involved in the definition of all the above DDF operators contains the 0-mode $k\cdot x$ of the coordinate field $k\cdot X$. The existence of this 0-mode implies that the above DDF operators do *not* commute with the (super-)Virasoro generators of the opposite chirality. In order to account for that one has to suitably multiply left- and right-moving DDF operators. The details of this will be discussed in §7.1.2.1 (p.140).

7.1.1.2 Pohlmeyer invariants. We now turn to the classical bosonic string and discuss the invartiants which have been studied by Pohlmeyer et al.

In the literature the invariance of the Pohlmeyer charges is demonstrated by the method of *Lax pairs*. But the same fact follows already from the well-known reparameterization invariance property of Wilson loops. To recall how this works for the classical bosonic string consider the following:

Denote the left- or rightmoving classical worldsheet fields in canonical language by $\mathcal{P}^\mu(\sigma)$, which have the canonical Poisson bracket

$$
\left[\mathcal{P}^\mu(\sigma), \mathcal{P}^\nu(\sigma')\right]_{\mathrm{PB}} = -\eta^{\mu\nu}\delta'(\sigma - \sigma')\;. \tag{7.17}
$$

The modes of the Virasoro constraints are

$$
L_m \equiv \frac{1}{2}\int d\sigma\, e^{-im\sigma}\eta_{\mu\nu}\mathcal{P}^\mu(\sigma)\,\mathcal{P}^\nu(\sigma) \tag{7.18}
$$

and the $\mathcal{P}(\sigma)$ transform with unit weight under their Poisson action:

$$
\left[L_m, \mathcal{P}^\mu(\sigma)\right]_{\mathrm{PB}} = \left(e^{-im\sigma}\mathcal{P}^\mu(\sigma)\right)'\;. \tag{7.19}
$$

This is all one needs to show that the *Pohlmeyer invariants* $Z^{\mu_1\cdots\mu_N}$ defined by

$$
Z^{\mu_1\cdots\mu_N}(\mathcal{P}) := \frac{1}{N}\int_0^{2\pi} d\sigma^1 \int_{\sigma^1}^{\sigma^1+2\pi} d\sigma^2 \cdots \int_{\sigma^{N-1}}^{\sigma^1+2\pi} d\sigma^N\, \mathcal{P}^{\mu_1}(\sigma^1)\,\mathcal{P}^{\mu_2}(\sigma^2)\cdots\mathcal{P}^{\mu_N}(\sigma^N) \tag{7.20}
$$

Poisson-commute with all the L_m.

The *proof* involves just a little combinatorics and algebra:

First note that if $F\big(\sigma^1, \sigma^2, \cdots, \sigma^N\big)$ is any function which is periodic with period 2π in each of its N arguments, the cyclically permuted path-ordered integral over F is equal to the integral used in (7.20)

$$\left[\int\limits_{0<\sigma^1<\sigma^2<\cdots<\sigma^N<2\pi} d^N\sigma \quad + \int\limits_{0<\sigma^N<\sigma^1<\cdots<\sigma^{N-1}<2\pi} d^N\sigma \quad + \int\limits_{0<\sigma^{N-1}<\sigma^N<\cdots<\sigma^{N-2}<2\pi} d^N\sigma \right] F(\sigma_1, \sigma_2, \cdots, \sigma_N)$$
$$= \int\limits_0^{2\pi} d\sigma^1 \int\limits_{\sigma^1}^{\sigma^1+2\pi} d\sigma^2 \cdots \int\limits_{\sigma^{N-1}}^{\sigma^1+2\pi} d\sigma^N \, F(\sigma_1, \sigma_2, \cdots, \sigma_N) \,. \tag{7.21}$$

(This follows by noting that while, for instance, σ^1 runs from 0 to 2π all other σ^i can be taken to run from σ^1 to $\sigma^1 + 2\pi$ while remaining in the correct order.)

This shows that the Pohlmeyer observables (7.20) are invariant under cyclic permutation of their indices. It can also be used to write their variation as

$$\begin{aligned} \delta Z^{\mu_1 \cdots \mu_N} = \frac{1}{N} \int\limits_0^{2\pi} d\sigma^1 \int\limits_{\sigma^1}^{\sigma^1+2\pi} d\sigma^2 \cdots \int\limits_{\sigma^{N-1}}^{\sigma^1+2\pi} d\sigma^N \Big(& \mathcal{P}^{\mu_1}\big(\sigma^1\big)\, \mathcal{P}^{\mu_2}\big(\sigma^2\big) \cdots \delta\mathcal{P}^{\mu_N}\big(\sigma^N\big) \\ & + \mathcal{P}^{\mu_N}\big(\sigma^1\big)\, \mathcal{P}^{\mu_1}\big(\sigma^2\big) \cdots \delta\mathcal{P}^{\mu_{N-1}}\big(\sigma^N\big) \\ & + \cdots \Big) \,, \end{aligned} \tag{7.22}$$

because we may cyclically permute the integration variables. But if one now sets $\delta\mathcal{P}^\mu(\sigma) = [L_m, \mathcal{P}^\mu(\sigma)]_{\text{PB}}$ one gets, using (7.19),

$$\begin{aligned} & \delta Z^{\mu_1 \cdots \mu_N} = \\ & \frac{1}{N} \int\limits_0^{2\pi} d\sigma^1 \int\limits_{\sigma^1}^{\sigma^1+2\pi} d\sigma^2 \cdots \int\limits_{\sigma^{N-2}}^{\sigma^1+2\pi} d\sigma^{N-1} \Big(\\ & \quad \xi\mathcal{P}^{\mu_N}\mathcal{P}^{\mu_1}\big(\sigma^1\big) \cdots \mathcal{P}^{\mu_{N-1}}\big(\sigma^{N-1}\big) - \mathcal{P}^{\mu_1}\big(\sigma^1\big) \cdots \xi\mathcal{P}^{\mu_{N-1}}\mathcal{P}^{\mu_N}\big(\sigma^{N-1}\big) \\ & \quad + \xi\mathcal{P}^{\mu_{N-1}}\mathcal{P}^{\mu_N}\big(\sigma^1\big) \cdots \mathcal{P}^{\mu_{N-2}}\big(\sigma^{N-1}\big) - \mathcal{P}^{\mu_N}\big(\sigma^1\big) \cdots \xi\mathcal{P}^{\mu_{N-2}}\mathcal{P}^{\mu_{N-1}}\big(\sigma^{N-1}\big) \\ & + \cdots \Big) \\ & = 0 \end{aligned} \tag{7.23}$$

(where we have written $\xi(\sigma) = e^{-im\sigma}$ for brevity). The contributions from the innermost integration cancel due to the cyclic permutation of integrands and integration variables. □

We note that the identity $[L_m, Z^{\mu_1 \cdots \mu_N}(\mathcal{P})]_{\text{PB}} = 0$ is just the infinitesimal version of the fact that the Pohlmeyer observables are invariant under *finite reparameterizations*

$$\mathcal{P}(\sigma) \mapsto \tilde{\mathcal{P}}(\sigma) \equiv R'(\sigma)\, \mathcal{P}(R(\sigma)) \tag{7.24}$$

induced by the *invertible* function R which is assumed to satisfy

$$R(\sigma + 2\pi) = R(\sigma) + 2\pi\,. \tag{7.25}$$

Indeed, we have the important relation

$$Z^{\mu_1\cdots\mu_N}(\mathcal{P}) \;=\; Z^{\mu_1\cdots\mu_N}\left(\tilde{\mathcal{P}}\right)\,, \tag{7.26}$$

which is at the heart of our derivation in §7.1.2.2 (p.144) that the Pohlmeyer invariants can be expressed in terms of DDF invariants

The *proof* of this involves just a simple change of variables in the integral:

$$\begin{aligned}
& Z^{\mu_1\cdots\mu_N}\left(\tilde{\mathcal{P}}\right) \\
= \quad & \frac{1}{N}\int\limits_0^{2\pi} d\sigma^1 \int\limits_{\sigma^1}^{\sigma^1+2\pi} d\sigma^2 \cdots \int\limits_{\sigma^{N-1}}^{\sigma^1+2\pi} d\sigma^N\, R'(\sigma^1)\, R'(\sigma^2)\cdots R'(\sigma^N)\, \mathcal{P}^{\mu_1}\left(R(\sigma^1)\right)\cdots\mathcal{P}^{\mu_N}\left(R(\sigma^N)\right) \\
\stackrel{\tilde\sigma^i\equiv R(\sigma^i)}{=} \quad & \frac{1}{N}\int\limits_{R(0)}^{R(2\pi)} d\tilde\sigma^1 \int\limits_{R(\sigma^1)}^{R(\sigma^1+2\pi)} d\tilde\sigma^2 \cdots \int\limits_{R(\sigma^{N-1})}^{R(\sigma^1+2\pi)} d\tilde\sigma^N\, \mathcal{P}^{\mu_1}\left(\tilde\sigma^1\right)\mathcal{P}^{\mu_2}\left(\tilde\sigma^2\right)\cdots\mathcal{P}^{\mu_N}\left(\tilde\sigma^N\right) \\
\stackrel{(7.25)}{=} \quad & \frac{1}{N}\int\limits_{R(0)}^{R(0)+2\pi} d\tilde\sigma^1 \int\limits_{\tilde\sigma^1}^{\tilde\sigma^1+2\pi} d\tilde\sigma^2 \cdots \int\limits_{\tilde\sigma^{N-1}}^{\tilde\sigma^1+2\pi} d\tilde\sigma^N\, \mathcal{P}^{\mu_1}\left(\tilde\sigma^1\right)\mathcal{P}^{\mu_2}\left(\tilde\sigma^2\right)\cdots\mathcal{P}^{\mu_N}\left(\tilde\sigma^N\right) \\
= \quad & Z^{\mu_1\cdots\mu_N}(\mathcal{P})\,.
\end{aligned} \tag{7.27}$$

□

$$[\mathcal{P}^\nu(\sigma), Z^{\mu_1\cdots\mu_N}]_{\mathrm{PB}} = \frac{N-2}{N}\left(\mathcal{P}^{\mu_1}(\sigma)\, Z^{\mu_2\cdots\mu_{N-1}}\eta^{\nu\mu_N} - Z^{\mu_1\cdots\mu_{N-2}}\mathcal{P}^{\mu_{N-1}}(\sigma)\,\eta^{\nu\mu_N}\right) + \text{cycl.} \tag{7.28}$$

(all indices transverse)

$$[A^\nu_m, Z^{\mu_1\cdots\mu_N}]_{\mathrm{PB}} = \frac{N-2}{N}\left(A^{\mu_1}_m Z^{\mu_2\cdots\mu_{N-1}}\eta^{\nu\mu_N} - Z^{\mu_1\cdots\mu_{N-2}}A^{\mu_{N-1}}_m\eta^{\nu\mu_N}\right) + \text{cycl.} \tag{7.29}$$

Finally, for the sake of completeness, we note the well-known fact that the Pohlmeyer invariants appear naturally as the Taylor-coefficients of *Wilson loops* along the string at constant worldsheet time. Let A_μ be a *constant but otherwise arbitrary* $\mathrm{GL}(N,\mathbf{c})$ connection on target space, then the Wilson loop around the string of this connection with respect to $\mathcal{P}$ is

$$\mathrm{Tr}\,\mathbf{P}\exp\left(\int\limits_0^{2\pi} d\sigma\, A_\mu\mathcal{P}^\mu(\sigma)\right) = \sum_{n=0}^{\infty} Z^{\mu_1\cdots\mu_n}(\mathcal{P})\;\mathrm{Tr}(A_{\mu_1}\cdots A_{\mu_2})\,, \tag{7.30}$$

where **P** denotes path-ordering.

This way of getting string "states" by means of Wilson lines of constant (but possibly large N) gauge connections along the string is intriguingly reminiscent of similar constructions used in the IIB Matrix Model (IKKT model) [149].

In the next sections the classical DDF invariants are described and it is shown how the Pohlmeyer invariants can be expressed in terms of these.

7.1.2 Classical bosonic DDF invariants and their relation to the Pohlmeyer invariants

The construction of classical DDF-like invariants for the *super*string, which is the content of §7.1.2.3 (p.145), is straightforward once the bosonic case is understood. The basic idea is very simple and shall therefore be given here first for the bosonic string, in order to demonstrate how §7.1.1.1 (p.134) and §7.1.1.2 (p.137) fit together.

7.1.2.1 Classical bosonic DDF invariants. In order to establish our notation and sign conventions we briefly list some definitions and relations which are in principle well known from elementary CFT but are rarely written out in the canonical language which we will need here.

So let $X(\sigma)$ and $P(\sigma)$ be canonical coordinates and momenta of the bosonic string with Poisson brackets

$$[X^\mu(\sigma), P_\nu(\kappa)]_{\mathrm{PB}} = \delta^\mu_\nu\, \delta(\sigma - \kappa)\ . \tag{7.31}$$

In close analogy to the CFT notation ∂X and $\bar\partial X$ we define

$$\mathcal{P}^\mu_\pm(\sigma) = \frac{1}{\sqrt{2T}} \left(P^\mu(\sigma) \pm T X'^\mu(\sigma)\right)\ . \tag{7.32}$$

(Here $T = 1/2\pi\alpha'$ is the string tension and we assume a trivial Minkowski background and shift all spacetime indices with $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, \cdots, 1)$.)

Their Poisson brackets are of course

$$\begin{aligned} \left[\mathcal{P}^\mu_\pm(\sigma), \mathcal{P}^\nu_\pm(\kappa)\right]_{\mathrm{PB}} &= \pm\eta^{\mu\nu}\delta'(\sigma - \kappa) \\ \left[\mathcal{P}^\mu_\pm(\sigma), \mathcal{P}^\nu_\mp(\kappa)\right]_{\mathrm{PB}} &= 0\,. \end{aligned} \tag{7.33}$$

From the mode expansion

$$\begin{aligned} \mathcal{P}^\mu_+(\sigma) &\equiv \frac{1}{\sqrt{2\pi}} \sum_m \tilde\alpha^\mu_m e^{-im\sigma} \\ \mathcal{P}^\mu_-(\sigma) &\equiv \frac{1}{\sqrt{2\pi}} \sum_m \alpha^\mu_m e^{+im\sigma} \end{aligned} \tag{7.34}$$

one finds the string oscillator Poisson algebra

$$[\alpha^\mu_m, \alpha^\nu_n]_{\mathrm{PB}} = -i\, m\, \eta^{\mu\nu} \delta_{m+n,0}\,, \tag{7.35}$$

as well as

$$[x^\mu, p^\nu]_{\mathrm{PB}} = \eta^{\mu\nu}\,, \tag{7.36}$$

where

$$\begin{aligned} x^\mu &\equiv \frac{1}{2\pi}\int X^\mu(\sigma)\; d\sigma \\ p^\mu &\equiv \int P^\mu(\sigma)\; d\sigma \;=\; \frac{1}{\sqrt{4\pi T}}\alpha_0 = \frac{1}{\sqrt{4\pi T}}\tilde{\alpha}_0\,. \end{aligned} \tag{7.37}$$

In terms of these oscillators the field X' reads

$$\begin{aligned} X'^\mu(\sigma) &= \frac{1}{\sqrt{2T}}\left(\mathcal{P}_+^\mu(\sigma) - \mathcal{P}_-^\mu(\sigma)\right) \\ &= \frac{1}{\sqrt{4\pi T}}\sum_{m=-\infty}^{\infty}\left(-\alpha_m^\mu + \tilde{\alpha}_{-m}^\mu\right)e^{+im\sigma} \end{aligned} \tag{7.38}$$

and hence the canonical coordinate field itself is

$$X^\mu(\sigma) = x^\mu + \frac{i}{\sqrt{4\pi T}}\sum_{m\neq 0}\frac{1}{m}\left(\alpha_m^\mu - \tilde{\alpha}_{-m}^\mu\right)e^{+im\sigma}\,. \tag{7.39}$$

Any field $A(\sigma)$ is said to have *classical conformal weight* $w(A)$ iff

$$[L_m, A(\sigma)] = e^{-im\sigma}A'(\sigma) + w(A)\,(e^{-im\sigma})'A(\sigma) \tag{7.40}$$

and is said to have classical conformal weight $\tilde{w}(A)$ iff

$$\left[\tilde{L}_m, A(\sigma)\right] = -e^{+im\sigma}A'(\sigma) - \tilde{w}(A)\,(e^{+im\sigma})'A(\sigma)\;, \tag{7.41}$$

where

$$\begin{aligned} L_m &\equiv \frac{1}{2}\int e^{-im\sigma}\mathcal{P}_-(\sigma)\cdot\mathcal{P}_-(\sigma) \;=\; \frac{1}{2}\sum_{k=-\infty}^{\infty}\alpha_{m-k}\cdot\alpha_k \\ \tilde{L}_m &\equiv \frac{1}{2}\int e^{+im\sigma}\mathcal{P}_+(\sigma)\cdot\mathcal{P}_+(\sigma) \;=\; \frac{1}{2}\sum_{k=-\infty}^{\infty}\tilde{\alpha}_{m-k}\cdot\tilde{\alpha}_k \end{aligned} \tag{7.42}$$

are the usual modes of the Virasoro generators.

The parts of $X(\sigma)$ which have $w = 0$ and $\tilde{w} = 0$, respectively, are

$$X_-^\mu(\sigma) \equiv x^\mu - \frac{\sigma}{4\pi T}p^\mu + \frac{i}{\sqrt{4\pi T}}\sum_{m\neq 0}\frac{1}{m}\alpha_m^\mu e^{+im\sigma} \tag{7.43}$$

and

$$X_+^\mu(\sigma) \equiv x^\mu + \frac{\sigma}{4\pi T}p^\mu + \frac{i}{\sqrt{4\pi T}}\sum_{m\neq 0}\frac{1}{m}\tilde{\alpha}_m^\mu e^{-im\sigma}\,. \tag{7.44}$$

This is checked by noticing the crucial property

$$\begin{aligned}\left(X_-^\mu\right)'(\sigma) &= -\frac{1}{\sqrt{2T}}\mathcal{P}_-^\mu(\sigma)\\ \left(X_+^\mu\right)'(\sigma) &= \frac{1}{\sqrt{2T}}\mathcal{P}_+^\mu(\sigma)\;.\end{aligned} \tag{7.45}$$

These weight 0 fields can now be used to construct "invariant oscillators", namely the classical DDF invariants:

To that end fix any lightlike vector field k on target space and consider the fields

$$R_\pm(\sigma) \equiv \pm\frac{4\pi T}{k\cdot p}\,k\cdot X_\pm(\sigma)\;. \tag{7.46}$$

The prefactor is an invariant and chosen so that

$$R_\pm(\sigma + 2\pi) = R_\pm(\sigma) + 2\pi\,. \tag{7.47}$$

Furthermore the derivative of $R_\pm$ is

$$R'_\pm(\sigma) = \frac{2\pi\sqrt{2T}}{k\cdot p}\,k\cdot\mathcal{P}_\pm(\sigma)\;. \tag{7.48}$$

It has been observed [150] that this derivative vanishes only on a subset of phase space of vanishing measure. This can be seen as follows:

The classical Virasoro constraints $\mathcal{P}_\pm^2 = 0$ say that $\mathcal{P}_\pm(\sigma)$ is lightlike. Because k is also lightlike this implies that $k\cdot\mathcal{P}_\pm(\sigma)$ vanishes iff $\mathcal{P}_\pm(\sigma)$ is parallel to k.

By writing $\mathcal{P}_\pm = \mathcal{P}_\pm^0\left[1, \mathcal{P}_\pm^i/\mathcal{P}_\pm^0\right]$ and noting that the spatial unit vector $e_\pm^i(\sigma) \equiv \mathcal{P}_\pm^i/\mathcal{P}_\pm^0$ is of weight $w = 0$ or $\tilde{w} = 0$ (while it Poisson commutes with the respective opposite Virasoro algebra), and hence transforms under the action of the Virasoro generators (which includes time evolution) as $e_\pm^i(\sigma) \to e_\pm^i(\sigma + f(\sigma))$, one sees that this condition is satisfied for some σ at some instance of time if and only if it is satisfied for some σ at any given time. In other words the time evolution of the string traces out trajectories in phase space which either have $\mathcal{P}_\pm$ parallel to k for some σ at *all times* or *never*.

In summary this means that except on the subset of phase space (of vanishing measure) of those trajectories where there exists a σ such that $k\cdot\mathcal{P}_\pm(\sigma) = 0$, the observables $R_\pm(\sigma)$ define invertible reparameterizations of the interval $[0, 2\pi)$, as considered in (7.25).

The above fact will be crucial below for expressing the Pohlmeyer invariants in terms of DDF invariants. For later usage let us introduce the notation $\mathbf{P}_k$ for the total phase space minus that set of vanishing measure:

$$\mathbf{P}_k \equiv \left\{(X(\sigma)\,, P(\sigma))_{\sigma\in(0,2\pi)} | k\cdot\mathcal{P}_\pm(\sigma) \neq 0\ \forall\sigma\right\}\;. \tag{7.49}$$

Now the classical DDF observables A_m^μ and $\tilde{A}_m^\mu$ of the closed bosonic string are finally defined (adapting the construction of (7.8) but using slighly different normalizations) by

$$\begin{aligned}A_m^\mu &\equiv \frac{1}{\sqrt{2\pi}}\int d\sigma\,\mathcal{P}_-^\mu(\sigma)\,e^{-imR_-(\sigma)}\\ \tilde{A}_m^\mu &\equiv \frac{1}{\sqrt{2\pi}}\int d\sigma\,\mathcal{P}_+^\mu(\sigma)\,e^{imR_+(\sigma)}\,.\end{aligned} \tag{7.50}$$

Note that the construction principle of these objects is essentially the same as that of the ordinary oscillators (7.34) except that the parameterization of the string used here differs from one point in phase space to the other.

Being integrals over fields of total weight $w = 1$ and $\tilde{w} = 1$, respectively, the DDF observables obviously Poisson-commute with their associated *half* of the Virasoro generators:

$$\begin{aligned} [L_m, A^\mu_n] &= 0 \\ \left[\tilde{L}_m, \tilde{A}^\mu_n\right] &= 0\,. \end{aligned} \tag{7.51}$$

But due to the coordinate 0-mode $\frac{2T}{kp}\, k\!\cdot\! x$ that enters the definition of $R_\pm$, the mixed Poisson-brackets do not vanish. In order to construct invariants one therefore has to split off this 0-mode and define the truncated observables

$$\begin{aligned} a^\mu_m &\equiv A^\mu_m e^{-im\frac{2T}{kp}\,kx} \\ \tilde{a}^\mu_m &\equiv A^\mu_m e^{-im\frac{2T}{kp}\,kx}\,. \end{aligned} \tag{7.52}$$

These now obviously have vanishing *mixed* Poisson brackets:

$$\begin{aligned} [L_m, \tilde{a}^\mu_n] &= 0 \\ \left[\tilde{L}_m, a^\mu_n\right] &= 0\,. \end{aligned} \tag{7.53}$$

Therefore classical DDF invariants which Poisson commute with *all* Virasoro constraints are obtained by forming products

$$D_{\{m_i,\tilde{m}_j\}} \equiv a^{\mu_1}_{m_1} \cdots a^{\mu_r}_{m_r}\, \tilde{a}^{\nu_1}_{\tilde{n}_1} \cdots a^{\nu_s}_{\tilde{m}_s} e^{iN\frac{2T}{kp}\,kx} \tag{7.54}$$

which satisfy the *level matching condition*:

$$\sum_i m_i \;=\; N \;=\; \sum_j \tilde{m}_j\,. \tag{7.55}$$

In order to see this explicitly write

$$\begin{aligned} \left[L_n, D_{\{m_i,\tilde{m}_j\}}\right]_{\mathrm{PB}} &= \underbrace{\left[L_n, a^{\mu_1}_{m_1} \cdots a^{\mu_r}_{m_r} e^{iN\frac{2T}{kp}\,kx}\right]_{\mathrm{PB}}}_{\overset{(7.51)}{=}0} \tilde{a}^{\nu_1}_{\tilde{n}_1} \cdots a^{\nu_s}_{\tilde{m}_s} + \\ &\quad + a^{\mu_1}_{m_1} \cdots a^{\mu_r}_{m_r} e^{iN\frac{2T}{kp}\,kx} \underbrace{\left[L_n, \tilde{a}^{\nu_1}_{\tilde{n}_1} \cdots a^{\nu_s}_{\tilde{m}_s}\right]_{\mathrm{PB}}}_{\overset{(7.53)}{=}0} \\ \left[\tilde{L}_n, D_{\{m_i,\tilde{m}_j\}}\right]_{\mathrm{PB}} &= \underbrace{\left[\tilde{L}_n, a^{\mu_1}_{m_1} \cdots a^{\mu_r}_{m_r}\right]_{\mathrm{PB}}}_{\overset{(7.53)}{=}0} \tilde{a}^{\nu_1}_{\tilde{n}_1} \cdots a^{\nu_s}_{\tilde{m}_s} e^{iN\frac{2T}{kp}\,kx} + \\ &\quad + a^{\mu_1}_{m_1} \cdots a^{\mu_r}_{m_r} \underbrace{\left[L_n, \tilde{a}^{\nu_1}_{\tilde{n}_1} \cdots a^{\nu_s}_{\tilde{m}_s} e^{iN\frac{2T}{kp}\,kx}\right]_{\mathrm{PB}}}_{\overset{(7.51)}{=}0}\,. \end{aligned} \tag{7.56}$$

This establishes the classical invariance of the DDF observables $D_{\{m_i,\tilde{m}_j\}}$. We next demonstrate how the Pohlmeyer invariants can be expressed in terms of DDF invariants.

7.1.2.2 Expressing Pohlmeyer invartiants in terms of DDF invariants. From the Fourier mode-like objects A^μ_m and $\tilde A^\mu_m$ one reobtains quasi-local fields[7] $\mathcal{P}^R_\pm$ by an inverse Fourier transformation:

$$\begin{aligned}
\mathcal{P}^R_-(\sigma) &\equiv \frac{1}{\sqrt{2\pi}}\sum_m A^\mu_m e^{+im\sigma} \;=\; \big((R_-)^{-1}\big)'(\sigma)\,\mathcal{P}^\mu\big((R_-)^{-1}(\sigma)\big)\\
\mathcal{P}^R_+(\sigma) &\equiv \frac{1}{\sqrt{2\pi}}\sum_m \tilde A^\mu_m e^{-im\sigma} \;=\; \big((R_+)^{-1}\big)'(\sigma)\,\mathcal{P}^\mu\big((R_+)^{-1}(\sigma)\big)\,.
\end{aligned} \tag{7.57}$$

This holds true on $\mathbf{P}_k$ (7.49) where we can use the fact that $R_\pm$ are invertible.

Comparison with (7.24) shows that these are just reparameterizations of the original local worldsheet fields $\mathcal{P}^\mu_\pm$, albeit with a reparameterization that varies from phase space point to phase space point, which is crucial for their invariance. But because the proof (7.27) of (7.26) involves only data available at a single point in phase space, it follows that for *every* invariant expression $F(\mathcal{P}_-)$ of the worldsheet fields $\mathcal{P}^\mu_-$ with $[L_m, F(\mathcal{P}_-)] = 0\,, \forall m$ we have

$$F(\mathcal{P}_-) \,=\, F(\mathcal{P}^R_-) \tag{7.58}$$

(on $\mathbf{P}_k$), and analogously for $\mathcal{P}_+$.

In summary we therefore obtain the following result:

On the restricted phase space $\mathbf{P}_\mathrm{k}$ (7.49) the classical Pohlmeyer invariants (7.20) can be expressed in terms of the classical DDF invariants (7.50) and the relation is

$$Z^{\mu_1\cdots\mu_N}(\mathcal{P}) \,=\, Z^{\mu_1\cdots\mu_N}(\mathcal{P}^R)\,, \tag{7.59}$$

where $\mathcal{P}$ is the ordinary worldsheet field (7.17), and $\mathcal{P}^R$ is the linear combination (7.57) of classical DDF observables.

This can be expressed in words also as follows: The Pohlmeyer invariants are left intact when replacing oscillators by respective DDF observables in their oscillator expansion ($\alpha^\mu_m \to A^\mu_m$, $\tilde\alpha^\mu_m \to \tilde A^\mu_m$). Note that the Pohlmeyer invariants are all of level 0 in the sense of (7.55) so that the level matching condition is trivially satisfied.

Because every polynomial in the DDF observables is consistently quantized by replacing A^μ_m and $\tilde A^\mu_m$ by the respective operators discussed in §7.1.1.1 (p.134), this yields a

[7] It is interesting to discuss these fields, and in particular their quantization, from the point of view of worldsheet (quantum) gravity:

Clearly the $\mathcal{P}^\mu_\pm(\sigma)$ are 'not physical' (do not Poisson commute with the constraints) because they evaluate the string's momentum and tension energy at a given value of the parameter σ, which of course has no physical relevance whatsoever. Heuristically, a physical observable may make recourse only to values of fields of the theory, not to values of auxiliary unphysical parameters. That is precisely the role played by the fields $R_\pm$. They allow to characterize a point of the string purely in terms of physical fields (string oscillations). Instead of asking: "What is the value of $\mathcal{P}_\pm$ at $\sigma = 3$?", we may ask the physically meaningful question: "What is the value of $\mathcal{P}_\pm$ at a point on the string where its configuration is such that $R_+ = 3$?" Quasi-local observables like the $\mathcal{P}^R_\pm$, or rather their absence, are related to old and well known issues of (quantum) gravity in higher dimensions, often referred to in the context of "*the problem of time*" [151].

It is maybe instructive to note how these issues are resolved here for the *worldsheet* theory of the relativistic string, a toy example for quantum gravity when regarded as a theory of 1+1 dimensional gravity. (Of course the string is rather more than a toy example for quantum gravity from the *target space* perspective.)

consistent quantization of the Pohlmeyer invariants.

Finally, by simply generalizing the DDF invariants to the superstring, equation (7.59) defines the generalization of the Pohlmeyer invariants to the superstring. This is discussed in the next subsection:

7.1.2.3 DDF and Pohlmeyer invariants for superstring. The additional fermionc fields on the classical superstring shall here be denoted by $\Gamma^\mu_\pm(\sigma)$, which are taken to be normalized so that their fermionic Poisson bracket reads

$$\begin{aligned}
\left\{\Gamma^\mu_\pm(\sigma), \Gamma^\nu_\pm(\kappa)\right\}_{\rm PB} &= \pm 2\eta^{\mu\nu}\delta(\sigma-\kappa)\\
\left\{\Gamma^\mu_\pm(\sigma), \Gamma^\nu_\mp(\kappa)\right\}_{\rm PB} &= 0\,.
\end{aligned} \tag{7.60}$$

The modes are of course

$$\begin{aligned}
b^\mu_r &\equiv \frac{i}{\sqrt{4\pi}}\int e^{-ir\sigma}\Gamma^\mu_-(\sigma)\\
\tilde b^\mu_r &= \frac{1}{\sqrt{4\pi}}\int e^{+ir\sigma}\Gamma^\mu_+(\sigma)
\end{aligned} \tag{7.61}$$

with non-vanishing brackets

$$\begin{aligned}
\{b^\mu_r, b^\nu_s\}_{\rm PB} &= -i\eta^{\mu\nu}\delta_{r+s,0}\\
\left\{\tilde b^\mu_r, \tilde b^\nu_s\right\}_{\rm PB} &= -i\eta^{\mu\nu}\delta_{r+s,0}\,,
\end{aligned} \tag{7.62}$$

and the fermionic part of the super Virasoro constraints are

$$\begin{aligned}
G_r &\equiv \frac{i}{\sqrt{2}}\int e^{-ir\sigma}\Gamma_-(\sigma)\cdot\mathcal{P}_-(\sigma)\;d\sigma &= \sum_{m=-\infty}^{\infty} b_{r+m}\cdot\alpha_{-m}\\
\tilde G_r &\equiv \frac{1}{\sqrt{2}}\int e^{+ir\sigma}\Gamma_+(\sigma)\cdot\mathcal{P}_+(\sigma)\;d\sigma &= \sum_{m=-\infty}^{\infty} \tilde b_{r+m}\cdot\tilde\alpha_{-m}\,.
\end{aligned} \tag{7.63}$$

The point is that we can entirely follow the constructions discussed in §7.1.1.1 (p.134) to get classical DDF invariants A^μ_m and B^μ_m which Poisson-commute with the full set of super Virasoro constraints. For instance in the R sector the DDF observable A^μ_m is

$$\begin{aligned}
A^\mu_m &\equiv \left\{G_0, \frac{i}{\sqrt{4\pi}}\int \Gamma^\mu_-(\sigma)\,e^{-imR_-(\sigma)}\right\}\\
&= \frac{1}{\sqrt{2\pi}}\int d\sigma\;\left(\mathcal{P}^\mu_-(\sigma) + im\frac{2\pi\sqrt{2T}}{k\cdot p}\Gamma^\mu_-(\sigma)\,k\cdot\Gamma_-(\sigma)\right)e^{-imR_-(\sigma)}\,.
\end{aligned} \tag{7.64}$$

By making the replacement $\alpha^\mu_m \to A^\mu_m$ in the ordinary Pohlmeyer invariant $Z^{\mu_1\cdots\mu_N}(\mathcal{P}_-)$ one obtains an object whose purely bosonic terms exactly coincide with the ordinary bosonic Pohlmeyer invariant and which furthermore has fermionic terms such that it super-Poisson-commutes with all super Virasoro constraints. This object is therefore obviously the superstring generalization of the ordinary Pohlmeyer invariant of the bosonic string.

7.2 Boundary States for D-Branes with Nonabelian Gauge Fields

In this subsection, which is taken from [35], we demonstrate a relation between two apparently unrelated aspects of superstrings: boundary states for nonabelian gauge fields and (super-)Pohlmeyer invariants.

On the one hand side superstring boundary states describing excitations of non-abelian gauge fields on D-branes are still the subject of investigations [152, 153, 30] and are of general interest for superstring theory, as they directly mediate between string theory and gauge theory.

On the other hand, studies of string quantization focusing on non-standard worldsheet invariants, the so-called Pohlmeyer invariants, done in [140, 141, 142, 143] and recalled in [139], were shown in [28, 29] to be related to the standard quantization of the string by way of the well-known DDF invariants. This raised the question whether the Pohlmeyer invariants are of any genuine interest in (super-)string theory as commonly understood.

Here it shall be shown that the (super-)Pohlmeyer invariants do indeed play an interesting role as boundary state deformation operators for non-abelian gauge fields, thus connecting the above two topics and illuminating aspects of both them.

A boundary state is a state in the closed string's Hilbert space constructed in such a way that inserting the vertex operator of that state in the path integral over the sphere reproduces the disk amplitudes for certain boundary conditions (D-branes) of the open string. In accord with the general fact that the worldsheet path integral insertions which describe background field excitations are exponentiations of the corresponding vertex operators, it turns out that the boundary states which describe gauge field excitations on the D-brane have the form of (generalized) Wilson lines of the gauge field along the closed string [154, 155, 152, 153, 30].

Long before these investigations, it was noted by Pohlmeyer [143], in the context of the classical string, that generalized Wilson lines along the closed string with respect to an auxiliary gauge connection on spacetime provide a "complete" set of invariants of the theory, i.e. a complete set of observables which (Poisson-)commute with all the Virasoro constraints.

Given these two developments it is natural to suspect that there is a relation between Pohlmeyer invariants and boundary states. Just like the DDF invariants (introduced in [147] and recently reviewed in [28]), which are the more commonly considered complete set of invariants of the string, commute with all the constraints and hence generate physical states when acting on the worldsheet vacuum, a consistently quantized version of the Pohlmeyer invariants should send boundary states of bare D-branes to those involving the excitation of a gauge field.

Indeed, up to a certain condition on the gauge field, this turns out to be true and works as follows:

If $X^\mu(\sigma)$ and $P_\mu(\sigma)$ are the canonical coordinates and momenta of the bosonic string, then $\mathcal{P}^\mu_\pm(\sigma) \equiv \frac{1}{\sqrt{2T}}(P_\mu(\sigma) \pm T\eta_{\mu\nu}X'^\nu(\sigma))$, (where T is the string's tension and a prime

denotes the derivative with respect to σ) are the left- and right-moving bosonic worldsheet fields for flat Minkowski background (in CFT context denoted by ∂X and $\bar{\partial} X$) and for any given constant gauge field A on target space the objects

$$W_{\pm}^{\mathcal{P}}[A] \equiv \mathrm{Tr}\,\mathrm{P}\exp\left(\int_0^{2\pi} d\sigma\, A\cdot\mathcal{P}_{\pm}(\sigma)\right) \tag{7.65}$$

(where Tr is the trace in the given representation of the gauge group's Lie algebra and P denotes path-ordering along σ) Poisson-commute with all Virasoro constraints. In fact the coefficients of $\mathrm{Tr}(A^n)$ in these generalized Wilson lines do so seperately, and these are usually addressed as the *Pohlmeyer invariants*, even though we shall use this term for the full object (7.65).

Fundamentally, the reason for this invariance is just the reparameterization invariance of the Wilson line, which can be seen to imply that (7.65) remains unchanged under a substitution of $\mathcal{P}$ with a reparameterized version of this field. In [28] it was observed that an interesting example for such a substitution is obtained by taking the ordinary DDF oscillators

$$A_m^{\mu} \propto \int_0^{2\pi} d\sigma\, \mathcal{P}_-^{\mu}(\sigma)\, e^{im\frac{4\pi T}{kp}\,kX_-(\sigma)} \tag{7.66}$$

(where k is a lightlike vector on target space, X_- is the left-moving component of X, p is the center-of-mass momentum, and an analogous expression exists for $\mathcal{P}_+$) and forming "quasi-local" invariants

$$\mathcal{P}_-^{R\mu}(\sigma) \equiv \frac{1}{\sqrt{2\pi}}\sum_{m=0}^{\infty} A_m^{\mu} e^{im\sigma} \tag{7.67}$$

from them.[8]

One finds

$$W^{\mathcal{P}}[A] = W^{\mathcal{P}^R}[A] \tag{7.68}$$

and since the quantization of the $\mathcal{P}^R$ in terms of DDF oscillators is well known, this gives a consistent quantization of the Pohlmeyer invariants. This is the quantization that we shall use here to study boundary states.

The above construction has a straightforward generalization to the superstring and this is the context in which the relation between the Pohlmeyer invariants and boundary states turns out to have interesting aspects, (while the bosonic case follows as a simple restriction, when all fermions are set to 0).

So we consider the supersymmetric extension of (7.66), which, by convenient abuse of notation, we shall also denote by A_m^{μ}:

$$A_m^{\mu} \propto \int_0^{2\pi} d\sigma \left(\mathcal{P}_-(\sigma) + im\frac{\pi\sqrt{2T}}{k\cdot p}k\cdot\Gamma_-(\sigma)\;\Gamma_-^{\mu}(\sigma)\right) e^{im\frac{4\pi T}{kp}\,kX_-(\sigma)}\,, \tag{7.69}$$

[8] We dare to use the same symbol A for the gauge field and for the DDF oscillators in order to comply with established conventions. The DDF oscillators will always carry a mode index m, however, and it should always be clear which object is meant.

where $\Gamma_\pm(\sigma)$ denote the fermionic superpartners of $\mathcal{P}_\pm$. From these we build again the objects (7.67) and finally $W^{\mathcal{P}^R}[A]$, which we address as the *super-Pohlmeyer* invariants.

Being constructed from the supersymmetric invariants $\mathcal{P}^R$, which again are built from (7.69), these manifestly commute with all of the super-Virasoro constraints. But in order to relate them to boundary states they need to be re-expressed in terms of the plain objects $\mathcal{P}$ and Γ. This turns out to be non-trivial and has some interesting aspects to it.

After these peliminaries we can state the first result to be reported here, which is

1. that on that subspace $\mathbf{P}_k$ of phase space where $k\cdot X_-$ is invertible as a function of σ (a condition that plays also a crucial role for the considerations of the bosonic DDF/Pohlmeyer relationship as discussed in [28]) the super-Pohlmeyer invariants built from (7.69) are equal to

$$W^{\mathcal{P}^R}[A]\Big|_{\mathbf{P}_k} = \mathrm{Tr}\,\mathrm{P}\exp\left(\int_0^{2\pi} d\sigma\ \left(iA_\mu + [A_\mu, A_\nu]\,\frac{k\cdot\Gamma\,\Gamma^\nu}{2k\cdot\mathcal{P}}\right)\mathcal{P}^\mu\right)\,, \qquad (7.70)$$

2. that this expression extends to an invariant on all of phase space precisely if the transversal components of A mutually commute,

3. and that in this case the above is equal to

$$Y[A] \equiv \mathrm{Tr}\,\mathrm{P}\exp\left(\int_0^{2\pi} d\sigma\ \left(iA_\mu\mathcal{P}^\mu + \frac{1}{4}\,[A_\mu, A_\nu]\,\Gamma^\mu\Gamma^\nu\right)\right)\,. \qquad (7.71)$$

The second result concerns the application of the quantum version of these observables to the bare boundary state $|\mathrm{D9}\rangle$ of a space-filling D9-brane (see for instance appendix A of [30] for a brief review of boundary state formalism and further literature). Denoting by $\mathcal{E}^\dagger(\sigma) = \frac{1}{2}\left(\Gamma_+(\sigma) + \Gamma_-(\sigma)\right)$ the differential forms on loop space (*cf.* section 2.3.1. of [30] and section 2.2 of [27] for the notation and nomenclature used here, and see [156] for a more general discussion of the loop space perspective) we find that for the above case of commuting transversal A the application of (7.71) to $|\mathrm{D9}\rangle$ yields

$$\begin{aligned}
&\mathrm{Tr}\,\mathrm{P}\exp\left(\int_0^{2\pi} d\sigma\ \left(iA_\mu\mathcal{P}^\mu + \frac{1}{4}(F_A)_{\mu\nu}\Gamma^\mu\Gamma^\nu\right)\right)|\mathrm{D9}\rangle \\
&= \mathrm{Tr}\,\mathrm{P}\exp\left(\int_0^{2\pi} d\sigma\ \left(-i\sqrt{\frac{T}{2}}A_\mu X'^\mu + \frac{1}{4}(F_A)_{\mu\nu}\mathcal{E}^{\dagger\mu}\mathcal{E}^{\dagger\nu}\right)\right)|\mathrm{D9}\rangle\,. \qquad (7.72)
\end{aligned}$$

which is, on the right hand side, precisely the boundary state describing a non-abelian gauge field on the D9 brane [152, 30] (for comparison one should rescale A as discussed in (7.114) below).

In summary this shows that and under which conditions the application of a quantized super-Pohlmeyer invariant to the boundary state of a bare D9 brane produces the boundary state describing a non-abelian gauge field excitation.

The structure of this paper closely follows the above outline:

First of all §7.2.1.1 (p.149) is concerned with the classical super-Pohlmeyer invariants and their expression in terms of local fields. Then §7.2.1.1 (p.149) discusses their cousins, the invariants of the general form (7.71). Both are related in §7.2.3 (p.153).

Then the quantization of the super-Pohlmeyer invariants is started in §7.2.4 (p.155). After an intermediate result concerning an operator ordering issue is treated in §7.2.5 (p.156) the quantum Pohlmeyer invariants are finally applied to the bare boundary state in §7.2.6 (p.158).

7.2.1 DDF operators, Pohlmeyer invariants and boundary states

7.2.1.1 Super-Pohlmeyer invariants. In [28] it was shown how from the classical DDF oscillators of the bosonic string one can construct *quasilocal* fields $\mathcal{P}^R$, which (Poisson-)commute with all the constraints and which, when used in place of X' in a Wilson line of a constant gauge field along the string, reproduce the Pohlmeyer invariants. It was mentioned that using the DDF oscillators of the superstring in this procedure leads to a generalization of the Pohlmeyer invariants to the superstring. Here we will work out the explicit form of the *super-Pohlmeyer invariants* obtained this way and point out that they are interesting in their own right.

Using the notation of [28] we denote by $\mathcal{P}^\mu(\sigma)$ the classical canonical left- or right-moving bosonic fields on the string, and by $\Gamma^\mu(\sigma)$ their fermionic partners, where the relation to the usual CFT notation is $\mathcal{P}^\mu \propto \partial X^\mu$ and $\Gamma^\mu \propto \psi^\mu$.

Our normalization is chosen such that the graded Poisson-brackets read

$$\begin{aligned}
[\Gamma^\mu(\sigma), \Gamma^\nu(\kappa)] &= -2\eta^{\mu\nu}\delta(\sigma-\kappa)\\
[\mathcal{P}^\mu(\sigma), \mathcal{P}^\nu(\kappa)] &= -\eta^{\mu\nu}\delta'(\sigma-\kappa)\ .
\end{aligned} \tag{7.73}$$

The classical bosonic DDF oscillators A^μ_m of the superstring are obtained by acting with the supercharge

$$G_0 = \frac{i}{\sqrt{2}}\int d\sigma\ \Gamma^\mu \mathcal{P}_\mu \tag{7.74}$$

(we concentrate on the Ramond sector for notational simplicity) on integrals over weight 1/2 fields:

$$\begin{aligned}
A^\mu_m &\equiv \left[G_0, \frac{i}{\sqrt{4\pi}}\oint d\sigma\ \Gamma^\mu e^{-imR}\right]\\
&= \frac{1}{\sqrt{2\pi}}\oint d\sigma\ \left(\mathcal{P}^\mu + im\frac{\pi\sqrt{2T}}{k\cdot p}k\cdot\Gamma\Gamma^\mu\right)e^{-imR}\,,
\end{aligned} \tag{7.75}$$

where

$$R(\sigma) \equiv -\frac{4\pi T}{k\cdot p}\,k\cdot X_\pm(\sigma) \tag{7.76}$$

and $p^\mu = \int_0^{2\pi} P^\mu(\sigma)$.

By construction, the A^μ_m super-Poisson-commute with all the constraints. From the A^μ_m quasi-local objects $\mathcal{P}^R$ are reobtained by Fourier transforming from the integral mode index m to the parameter σ:

$$\begin{aligned}\mathcal{P}^R(\sigma) &\equiv \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} A_n\, e^{in\sigma} \\ &= \int_0^{2\pi} d\tilde\sigma\, \left(\mathcal{P}(\tilde\sigma)\, \delta(R(\tilde\sigma)-\sigma) + \frac{i\pi\sqrt{2T}}{k\cdot p} k\cdot\Gamma(\tilde\sigma)\,\Gamma(\tilde\sigma)\, \frac{\partial}{\partial\sigma}\delta(R(\tilde\sigma)-\sigma)\right) . \end{aligned} \tag{7.77}$$

The role of the DDF oscillators played here is the derivation of this expression. Their invariance was rather easy to enforce and check, but by taking combinations of them as in (7.77) and further constructions below, we can now build objects which are necessarily still invariants, but whose invariance is much less obvious.

Since the DDF oscillators A^μ_m won't be explicitly needed anymore in the following, we take the liberty to reserve the letter A from now on to describe a gauge connection on target space. We shall be interested in the Wilson line

$$W^{\mathcal{P}^R}[A] \equiv \mathrm{Tr}\,\mathrm{P}\exp\left(i \int_0^{2\pi} d\sigma\; A\cdot\mathcal{P}^R(\sigma)\right) \tag{7.78}$$

with respect to this gauge connection A, constructed using the "generalized tangent vector" $\mathcal{P}^R$ which plays the role of the true tangent vector X' found in ordinary Wilson lines. Because this object follows in spirit closely the construction principle of the bosonic Pohlmeyer invariants, and because its bosonic component coincides with the purely bosonic Pohlmeyer invariant, we shall here address it as the *super-Pohlmeyer invariant*. In the following a form of this object in terms of the original local fields $\mathcal{P}$ and Γ is derived, which will illuminate its relation to supersymmetric boundary states.

The integrand of (7.78) can be put in a more insightful form by means of a couple of manipulations:

Following the development in [28] (*cf.* equation (2.43)) we now temporarily restrict attention to the subspace $\mathbf{P}_k$ of phase space on which the function R is invertible, in which case it is, by construction, 2π-periodic. On that part of phase space (and only there) the integral in (7.77) can be evaluated to yield

$$\mathcal{P}^R(\sigma)\big|_{\mathbf{P}_k} = \left(R^{-1}\right)'(\sigma)\, \mathcal{P}\left(R^{-1}(\sigma)\right) + \frac{\pi i\sqrt{2T}}{k\cdot p}\frac{\partial}{\partial\sigma}\left(\left(R^{-1}\right)'(\sigma)\, k\cdot\Gamma\left(R^{-1}(\sigma)\right)\Gamma\left(R^{-1}(\sigma)\right)\right) . \tag{7.79}$$

The first term is known from the bosonic theory (equation (2.51) in [28]). The second term involves the fermionic correction due to supersymmetry, and its remarkable property is that it is a total σ-derivative. This means that when $\mathcal{P}^R$ is inserted in a multi-integral as they appear in (7.78), the fermionic term will produce boundary terms and hence coalesce with neighbouring integrands.

Before writing this down in more detail first note that due to k being a null vector the fermionic terms can never coalesce with themselves, because of

$$
\begin{aligned}
&\left(R^{-1}\right)' k\cdot\Gamma\left(R^{-1}\right) A\cdot\Gamma\left(R^{-1}\right)(\sigma)\,\frac{\partial}{\partial\sigma}\Big(\left(R^{-1}\right)' k\cdot\Gamma\left(R^{-1}\right) A\cdot\Gamma\left(R^{-1}\right)\Big)(\sigma)\\
&= \frac{1}{2}\frac{\partial}{\partial\sigma}\underbrace{\Big(\left(R^{-1}\right)' k\cdot\Gamma\left(R^{-1}\right) A_\mu\Gamma_-^\mu\left(R^{-1}\right)\Big)^2(\sigma)}_{=0}\\
&= 0\,.
\end{aligned}
\tag{7.80}
$$

This vanishing result depends on the Grassmann properties of the classical fermions Γ, which we are dealing with here. The generalization of the present development to the quantum theory requires more care and is dealt with below.

Using (7.80) a little reflection shows that, when the total derivative terms in (7.78) are all integrated over and coalesced at the integration bounds with the neighbouring terms $iA\cdot\mathcal{P}$, this yields

$$
\begin{aligned}
&\mathrm{Tr}\,\mathrm{P}\exp\left(i\int_0^{2\pi} d\sigma\; A\cdot\mathcal{P}^R(\sigma)\right)\Big|_{\mathbf{P}_k}\\
&= \mathrm{Tr}\,\mathrm{P}\exp\left(\int_0^{2\pi} d\sigma\;\left(iA_\mu + [A_\mu, A_\nu]\,\frac{\pi\sqrt{2T}}{k\cdot p}(R^{-1})'(\sigma)\,k\cdot\Gamma\left(R^{-1}(\sigma)\right)\;\Gamma^\nu\left(R^{-1}(\sigma)\right)\right)(R^{-1})'(\sigma)\,\mathcal{P}^\mu\left(R^{-1}(\sigma)\right)\right)\,.
\end{aligned}
\tag{7.81}
$$

This expression simplifies drastically when a change of variable $\tilde\sigma \equiv R^{-1}(\sigma)$ is performed in the integral, as in (2.23) of [28]:

$$
\cdots = \mathrm{Tr}\,\mathrm{P}\exp\left(\int_0^{2\pi} d\tilde\sigma\;\left(iA_\mu + [A_\mu, A_\nu]\,\frac{\pi\sqrt{2T}}{k\cdot p}(R^{-1})'(R(\tilde\sigma))\,k\cdot\Gamma(\tilde\sigma)\;\Gamma^\nu(\tilde\sigma)\right)\mathcal{P}^\mu(\tilde\sigma)\right)\,.
\tag{7.82}
$$

The fermionic term further simplifies by using $(R^{-1})'(R(\tilde\sigma)) = 1/R'(\tilde\sigma)$ and then equation (2.42) of [28], which gives $(R^{-1})'(R(\tilde\sigma)) = \frac{kp}{2\pi\sqrt{2T}}\frac{1}{k\mathcal{P}}$. This way the above is finally rewritten as

$$
W^{\mathcal{P}^R}[A]\Big|_{\mathbf{P}_k} = \mathrm{Tr}\,\mathrm{P}\exp\left(\int_0^{2\pi} d\sigma\;\left(iA_\mu + [A_\mu, A_\nu]\,\frac{k\cdot\Gamma\,\Gamma^\nu}{2k\cdot\mathcal{P}}\right)\mathcal{P}^\mu\right)\,.
\tag{7.83}
$$

This is the advertized explicit form of the super-Pohlmeyer invariant in terms of local fields, when restricted to $\mathbf{P}_k$.

The right hand side extends to an observable on all of phase space in the obvious way and it is of interest to study if this extension is still an invariant. This is the content of the following subsections.

7.2.2 Another supersymmetric extension of the bosonic Pohlmeyer invariants

We address the objects (7.78) as super-Pohlmeyer invariants, because they are obtained from the bosonic Pohlmeyer invariants written in the form $\mathrm{Tr}\,\mathrm{P}\exp\left(\int_0^{2\pi} d\sigma\, A\cdot\mathcal{P}^R(\sigma)\right)$ of

equation (2.52) of [28] by replacing the bosonic *quasi-local* invariants $\mathcal{P}^R$ by their supersymmetric version (7.77). In this sense this supersymmetric extension is *local*, or rather "quasi-local", since the $\mathcal{P}^R$ are. But it turns out that there is another fermionic extension of the bosonic Pohlmeyer invariant $\mathrm{Tr}\,\mathrm{P}\exp\left(\int_0^{2\pi} d\sigma\; A\cdot\mathcal{P}(\sigma)\right)$ which Poisson-commutes with all the super-Virasoro generators, and which is not local in this sense, namely

$$Y[A] \equiv \mathrm{Tr}\,\mathrm{P}\exp\left(\int_0^{2\pi} d\sigma\; \left(iA_\mu \mathcal{P}^\mu(\sigma) + \frac{1}{4}\left[A_\mu, A_\nu\right]\Gamma^\mu(\sigma)\,\Gamma^\nu(\sigma)\right)\right) . \tag{7.84}$$

Here the integrand itself does not Poisson-commute with the supercharge G_0, but $Y[A]$ as a whole does. (This can easily be generalized even to non-constant A, but we will here be content with writing down all expression for the case of constant A. Non-constant A will be discussed in the context of the quantum theory further below.)

Invariance under the bosonic Virasoro generators is immediate, because the integrand has unit weight. All that remains to be checked is hence

$$[G_0, Y[A]] = 0 . \tag{7.85}$$

Proof: This is best seen by following the logic involved in the derivation of equation (3.8) in [30]: There are terms coming from $[G_0, iA\cdot\mathcal{P}(\sigma)] \propto iA\cdot\Gamma'(\sigma)$ which coalesce at the integration boundary with $iA\mathcal{P}$ to give $-\left[A_\mu, A_\nu\right]\Gamma^\mu\mathcal{P}^\nu$. This cancels with the contribution from $\left[G_0, \frac{1}{4}\left[A_\mu, A_\nu\right]\Gamma^\mu\Gamma^\nu\right] \propto \left[A_\mu, A_\nu\right]\Gamma^\mu\mathcal{P}^\nu$. (Here we write $\propto$ only as a means to ignore the irrelevant global prefactor $i/\sqrt{2}$ in (7.74).) Moreover, there is coalescence of $A\cdot\Gamma'$ with $\left[A_\mu, A_\nu\right]\Gamma^\mu\Gamma^\nu$ which yields $\left[A_\kappa, \left[A_\mu, A_\nu\right]\right]\Gamma^\kappa\Gamma^\mu\Gamma^\nu = 0$, so that everything vanishes. This establishes the full invariance of $Y[A]$ under the super-Virasoro algebra. □

With this insight in hand, one can make a curious observation. Write $A_+ \equiv k\cdot A$ and consider the special case where all transversal components of A together with the additional lightlike component A_- mutually commute

$$[A_i, A_j] = 0\,, \;\; \forall\, i,j \neq + . \tag{7.86}$$

Then

$$\begin{aligned} \left[A_\mu, A_\nu\right]\frac{k\cdot\Gamma\Gamma^\nu}{2k\cdot\mathcal{P}}\mathcal{P}^\mu &= \frac{1}{2}\left[A_+, A_i\right]\Gamma^+\Gamma^i \\ &= \frac{1}{4}\left[A_\mu, A_\nu\right]\Gamma^\mu\Gamma^\nu . \end{aligned} \tag{7.87}$$

Comparison of (7.83) with (7.84) hence shows that in this case the super-Pohlmeyer invariant (7.83) and the invariant (7.84) coincide:

$$[A_i, A_j] = 0\,, \;\; \forall\, i,j \neq + \Rightarrow\; W^{\mathcal{P}^R}[A]\Big|_{\mathbf{P}_k} = Y[A] . \tag{7.88}$$

So in particular in the case (7.86) the extension of the right hand side of (7.83) to all of phase space is still an invariant.

Comparing (7.83) with equation (3.14) of [30] it is obvious, and will be discussed in more detail below, that $Y[A]$ must somehow be closely related to the boundary deformation operator describing non-abelian A-field excitations. Together with (7.88) this gives a first indication of how super-Pohlmeyer invariants give insight into boundary states of the superstring.

Before discussing this in more detail the next section investigates the most general condition under which the extension of the right hand side of (7.83) to all of phase space is still an invariant. It turns out that there are other cases besides (7.86).

7.2.3 Invariance of the extension of the restricted super-Pohlmeyer invariants

For the bosonic string the constraint $\mathcal{P}\cdot\mathcal{P} = 0$, which says that $\mathcal{P}$ is a null vector in target space, ensured that the invertibility of R was preserved by the evolution generated by the constraints (*cf.* the discussion on p.12 of [28]).

The same is no longer true for the superstring, where we schematically have $\mathcal{P}\cdot\mathcal{P}+\Gamma\cdot\Gamma' = 0$, instead. It follows that we cannot expect the extension of the right hand side of (7.83) to all of (super-)phase space to super-Poisson commute with all the constraints, since the flow induced by the constraints will in general leave the subspace $\mathbf{P}_k$. Only for the bosonic string does the flow induced by the constraints respect $\mathbf{P}_k$.

Notice that this is not in contradiction to the above result that on $\mathbf{P}_k$ the super-Pohlmeyer invariant (7.78) (which by construction super-Poisson commutes with all the constraints) coincides with (7.83). Two functions which conincide on a subset of their mutual domains need not have coinciding derivatives at these points.

First of all one notes that the invariance under the action of the bosonic constraints is still manifest in (7.83). Because the integrand still has unit weight one checks this simply by using the same reasoning as in equation (2.19) of [28].

But the result of super-Poisson commuting with the supercharge G_0 is rather non-obvious. A careful calculation shows that the result vanishes if and only if

$$[A_i, A_j] = 0\,, \ \ \forall\, i,j \neq + \tag{7.89}$$

or

$$k\cdot\Gamma' = 0 = k\cdot\mathcal{P}'\,. \tag{7.90}$$

The first condition is that already discussed in §7.2.2 (p.151). The second condition is nothing but the defining condition of *lightcone gauge* on the worldsheet.

Notice that these two conditions are very different in character. When the first (7.89) is satisfied it means that the extension of the right hand side of (7.83) to all of phase space is indeed an honest invariant. When the first condition is not satisfied then the extenstion of the right hand side of (7.89) to all of phase space is simply not an invariant. Still, it is an object whose Poisson-commutator with the super-Virasoro constraints vanishes on that part of phase space where (7.90) holds.

We now conclude this subsection by giving the detailed *proof* for the above two conditions.

Proof:

First consider the terms of fermionic grade 1. These are contributed by

$$[G_0, A\cdot\mathcal{P}] \propto A\cdot\Gamma' \tag{7.91}$$

as well as

$$[A_\mu, A_\nu]\,\frac{[G_0, k\cdot\Gamma]\,\Gamma^\nu\mathcal{P}^\mu}{2k\cdot\mathcal{P}} \propto -\,[A_\mu, A_\nu]\,\Gamma^\nu\mathcal{P}^\mu\,. \tag{7.92}$$

The other remaining fermionic contraction does not contribute, due to

$$[A_\mu, A_\nu]\,[G_0, \Gamma^\nu]\,\mathcal{P}^\mu \propto -2\,[A_\mu, A_\nu]\,\mathcal{P}^\nu\mathcal{P}^\mu = 0\,. \tag{7.93}$$

In the path ordered integral the terms (11.49) appear as

$$\begin{aligned}
&\cdots iA\cdot\mathcal{P}(\sigma_{i-1})\int_{\sigma_{i-1}}^{\sigma_{i+1}} iA\cdot\Gamma'(\sigma_i)\;\,d\sigma_i\;iA\cdot\mathcal{P}(\sigma_{i+1})\cdots\\
&= \cdots((A\cdot\mathcal{P}A\cdot\Gamma)(\sigma_{i-1})\,iA\cdot\mathcal{P}(\sigma_{i+1}) - iA\cdot\mathcal{P}(\sigma_{i-1})\,(A\cdot\Gamma A\cdot\mathcal{P})(\sigma_{i+1}))\cdots\,.
\end{aligned} \tag{7.94}$$

(This is really a special case of the general formula (3.8) in [30].) This way the term

$$iA_\mu\Gamma^\mu\,iA_\nu\mathcal{P}^\nu - iA_\mu\mathcal{P}^\nu\,iA_\mu\Gamma^\mu = [A_\mu, A_\nu]\,\Gamma^\nu\mathcal{P}^\mu \tag{7.95}$$

is produced, and it cancels precisely with (7.92).

This verifies that there are no terms of grade 1.

Now consider the remaining terms of grade 3. It is helpful to write

$$[A_\mu, A_\nu]\,\frac{k\cdot\Gamma\Gamma^\nu}{2k\cdot\Gamma}\mathcal{P}^\mu = \frac{1}{2}\,[A_+, A_i]\,\Gamma^+\Gamma^i + [A_i, A_j]\,\frac{k\cdot\Gamma\Gamma^j}{2k\cdot\mathcal{P}}\mathcal{P}^i\,. \tag{7.96}$$

The first term on the right hand side gives nothing of grade 3 when Poisson-commuted with G_0. The second term however gives rise to

$$\begin{aligned}
\left[G_0, [A_i, A_j]\,\frac{k\cdot\Gamma\,\Gamma^j}{2k\cdot\mathcal{P}}\mathcal{P}^i\right] &= [A_i, A_j]\left(\frac{k\cdot\Gamma\,\Gamma^j}{2k\cdot\mathcal{P}}\Gamma'^i - \frac{k\cdot\Gamma\,\Gamma^j}{2(k\cdot\mathcal{P})^2}k\cdot\Gamma'\right) + \text{terms already considered}\\
&= [A_i, A_j]\left(\frac{k\cdot\Gamma}{4k\cdot\mathcal{P}}(\Gamma^j\Gamma^i)' - \frac{k\cdot\Gamma\,\Gamma^j}{2(k\cdot\mathcal{P})^2}k\cdot\Gamma'\right) + \text{terms already considered}\\
&= [A_i, A_j]\left(\frac{k\cdot\Gamma}{4k\cdot\mathcal{P}}\Gamma^j\Gamma^i\right)' + \alpha + \text{terms already considered}\,,
\end{aligned} \tag{7.97}$$

where we have abbreviated with

$$\alpha \equiv -\,[A_i, A_j]\left(\left(\frac{k\cdot\Gamma}{2k\cdot\mathcal{P}}\right)'\Gamma^j\Gamma^i + \frac{k\cdot\Gamma\,\Gamma^j}{2(k\cdot\mathcal{P})^2}k\cdot\Gamma'\right) \tag{7.98}$$

two terms which will *not* cancel with anything in the following. (Notice that they are proportional to σ-derivatives of longitudinal objects (along k).)

The remaining first term on the right hand side of (7.97) coalesces with $iA_+\mathcal{P}^+$ to yield $\frac{i}{4}\left[A_+,[A_i,A_j]\right]\Gamma^+\Gamma^i\Gamma^j$. This cancels against the coalescence of (11.49) with the first term on the right hand side of (7.96) which gives the term $\frac{i}{2}\left[A_j,[A_+,A_i]\right]\Gamma^j\Gamma^+\Gamma^i$, because together they become the longitudinal component of the exterior covariant derivative of the field strength of A, which vanishes. The transversal component of this exterior derivative of the field strength appears in the remaining terms:

First there is the remaining coalescence of (11.49) with the second term on the right hand side of (7.96), which yields $i\left[A_k,[A_i,A_j]\right]\frac{k\Gamma}{2k\mathcal{P}}\Gamma^k\Gamma^j\mathcal{P}^i$. Together with the remaining coalescence of the first term on the right of (7.97) with the transversal $iA_j\mathcal{P}^j$ which gives rise to $i\left[A_k,[A_i,A_j]\right]\frac{k\Gamma}{4k\mathcal{P}}\Gamma^i\Gamma^j\mathcal{P}^k$ one gets something proportional to $\Big[G_0,\underbrace{[A_k,[A_i,A_j]]\,\Gamma^k\Gamma^i\Gamma^j}_{=0}\Big] = 0$, which vanishes because it involves the transversal part of the gauge covariant exterior derivative of the field strength of A.

In summary, the only terms that remain are those of (7.98). When the σ-derivative is written out this are three terms which have to vanish seperately, because they contain different combinations of fermions. Clearly they vanish precisely if (7.89) or (7.90) are satisfied. This completes the proof. $\square$

.

7.2.4 Quantum super-Pohlmeyer invariants

The DDF-invariants (7.75) are, as discussed in equation (2.12) of [28], still invariants after quantization in terms of DDF oscillators. If we take the liberty to denote the quantized objects $\mathcal{P}$ and Γ by the same symbols as their classical counterparts, then the only thing that changes in the notation of the above sections is that the canonical super-commutation relations (7.73) pick up an imaginary factor

$$[\mathcal{P}^\mu(\sigma),\mathcal{P}(\kappa)] = -i\eta^{\mu\nu}\delta'(\sigma-\kappa)\,. \tag{7.99}$$

This again introduces that same factor in the second term of (7.75) and similarly in the following expressions.

The quantization of the super-Pohlmeyer invariant (7.78) is a trivial consequence of the quantization of the DDF invariants that it is built from, and, with that imaginary unit taken care of, its restriction (7.83) to the case where R is invertible reads

$$\mathrm{Tr}\,\mathrm{P}\exp\left(\int\limits_0^{2\pi} d\sigma\;\left(iA_\mu + \frac{i}{2}\left[A_\mu,A_\nu\right]\frac{k\cdot\Gamma\Gamma^\nu}{k\cdot\mathcal{P}}\right)\mathcal{P}^\mu\right)\,. \tag{7.100}$$

Noting that our A is taken to be hermitian and that hence the gauge field strength is

$$\begin{aligned}
F_A &= -i(d+iA)^2\\
&= dA + iA\wedge A\\
&= \left(\partial_{[\mu}A_{\nu]} + \frac{i}{2}\left[A_\mu,A_\nu\right]\right)dx^\mu\wedge dx^\nu\\
&= \frac{1}{2}(F_A)_{\mu\nu}dx^\mu\wedge dx^\nu
\end{aligned} \tag{7.101}$$

the second term in the integrand is related to the field strength as in

$$\cdots = \mathrm{Tr}\,\mathrm{P}\exp\left(\int\limits_0^{2\pi} d\sigma\ \left(iA_\mu + \frac{1}{2}(F_A)_{\mu\nu}\frac{k\cdot\Gamma\Gamma^\nu}{k\cdot\mathcal{P}}\right)\mathcal{P}^\mu\right)\,. \tag{7.102}$$

In the case $[A_i, A_j] = 0$ (7.86) we hence obtain the quantized version of (7.84) in the form

$$Y[A] = \mathrm{Tr}\,\mathrm{P}\exp\left(\int_0^{2\pi} d\sigma\ \left(iA_\mu\mathcal{P}^\mu + \frac{1}{4}(F_A)_{\mu\mu}\Gamma^\mu\Gamma^\nu\right)\right)\,. \tag{7.103}$$

While the quantized super-Pohlmeyer invariant, being constructed from invariant DDF operators, is itself a quantum invariant in that it commutes with all the super-Virasoro constraints, the proof in §7.2.3 (p.153) of the invariance condition of the restricted and then extended form (7.83) receives quantum corrections. In its classical version the proof makes use of the Grassmann property of the fermions Γ. Quantumly there will be diverging contractions in products of Γs which not only prevent the application of the proof to the quantum theory but also make the expression (7.100) ill defined without some regularization prescription.

This application of (7.103) to a bare boundary states is the content of §7.2.6 (p.158). But before coming to that a technicality needs to be discussed, which is done in the next section.

7.2.5 On an operator ordering issue in Wilson lines along the closed string

For applying a generalized Wilson line of the kind discussed above to any string state, it is helpful to understand how the operators in the Wilson line can be commuted past each other to act on the state on the right. It turns out that under a certain condition, which is fulfilled in the cases we are interested in, the operators can be freely commuted. This works as follows:

A generalized Wilson line of the form

$$W^{\mathcal{P}}[A] = \mathrm{Tr}\,\mathrm{P}\exp\left(\int\limits_0^{2\pi} A\cdot\mathcal{P}(\sigma)\ d\sigma\right) \tag{7.104}$$

with even graded $\mathcal{P}$ (which could be the $\mathcal{P}$ or $\mathcal{P}^R$ of the previous sections but also more general objects) breaks up like

$$W^{\mathcal{P}}[A] = \sum_{n=0}^{\infty} Z^{\mu_1\cdots\mu_n}\mathrm{Tr}(A_{\mu_1}\cdots A_{\mu_2}) \tag{7.105}$$

into iterated integrals

$$Z^{\mu_1\cdots\mu_N} = \frac{1}{N}\left[\int\limits_{0<\sigma^1<\sigma^2<\cdots<\sigma^N<2\pi} d^N\sigma + \int\limits_{0<\sigma^N<\sigma^1<\cdots<\sigma^{N-1}<2\pi} d^N\sigma + \int\limits_{0<\sigma^{N-1}<\sigma^N<\cdots<\sigma^{N-2}<2\pi} d^N\sigma + \cdots\right]\mathcal{P}^{\mu_1}(\sigma^1)\cdots\mathcal{P}^{\mu_N}(\sigma^N)\,. \tag{7.106}$$

In equation (2.17) of [28] if was noted that the integration domain can equivalently be written as

$$Z^{\mu_1\cdots\mu_N} = \frac{1}{N}\int_0^{2\pi} d\sigma^1 \int_{\sigma^1}^{\sigma^1+2\pi} d\sigma^2 \cdots \int_{\sigma^{N-1}}^{\sigma^1+2\pi} d\sigma^N \, \mathcal{P}^{\mu_1}\big(\sigma^1\big)\,\mathcal{P}^{\mu_2}\big(\sigma^2\big)\cdots\mathcal{P}^{\mu_N}\big(\sigma^N\big) \,. \tag{7.107}$$

This is seen by simply replacing all $\sigma^i < \sigma^1$ for $i > 1$ by $\sigma^i + 2\pi$. Due to the periodicity of $\mathcal{P}$ this does not change the value of the integral but yields the integration bounds used in (7.107).

The reason why this is recalled here is that a slight generalization of this fact will be needed in the following. Namely for any integer M with $1 < M < N$ one can obviously more generally write

$$Z^{\mu_1\cdots\mu_N} = \int_0^{2\pi} d\sigma^1 \int_{\sigma^1}^{\sigma^1+2\pi} d\sigma^M \int_{\sigma^1}^{\sigma^M} d\sigma^2 \cdots \int_{\sigma^{M-2}}^{\sigma^M} d\sigma^{M-1} \int_{\sigma^M}^{\sigma^1+2\pi} d\sigma^{M+1} \cdots \int_{\sigma^{N-1}}^{\sigma^1+2\pi} d\sigma^N \, \mathcal{P}^{\mu_1}\big(\sigma^1\big)\,\mathcal{P}^{\mu_2}\big(\sigma^2\big)\cdots\mathcal{P}^{\mu_N}\big(\sigma^N\big) \,. \tag{7.108}$$

Equation (7.107) follows as the special case with $M = 2$.

The motivation for these considerations is the following:

Classically, the $\mathcal{P}$ commute among each other. Therefore the ordering of the $\mathcal{P}$ in the integrand makes no difference, only the combination of spacetime index μ_i with integration variable σ^i does.

Here we want to note that this remains true at the quantum level *if*

$$[\mathcal{P}(\sigma), \mathcal{P}(\kappa)] \propto \delta'(\sigma - \kappa) \,. \tag{7.109}$$

This is readily seen by commuting $\mathcal{P}\big(\sigma^1\big)$ with $\mathcal{P}\big(\sigma^M\big)$ in (7.108). The result has the form

$$\begin{aligned}
\int_0^{2\pi} d\sigma^1 \int_{\sigma^1}^{\sigma^1+2\pi} d\sigma^M \, \delta'\big(\sigma^1 - \sigma^M\big)\, F\big(\sigma^1, \cdots, \sigma^N\big) &= \int_0^{2\pi} d\sigma^1 \int_{\sigma^1}^{\sigma^1+2\pi} d\sigma^M \, \delta\big(\sigma^1 - \sigma^\kappa\big)\, \frac{\partial}{\partial\sigma^M} F\big(\sigma^1, \cdots, \sigma^N\big) \\
&= \int_{\sigma^1}^{\sigma^1+2\pi} d\sigma^M \, \delta\big(\sigma^1 - \sigma^\kappa\big)\, \frac{\partial}{\partial\sigma^M} F\big(\sigma^1, \cdots, \sigma^N\big) \\
&= 0 \,,
\end{aligned} \tag{7.110}$$

so that all resulting commutator terms vanish. Every other commutator can be obtained by using the cyclic invariance in the integration variables.

More generally, any two (even graded, periodic) objects $A(\sigma)$, $B(\kappa)$ in the integrand of an iterated integral of the form (7.106) whose commutator is proportional to $[A(\sigma), B(\kappa)] \propto \delta'(\sigma - \kappa)$ can be commuted past each other in the Wilson line without affecting the value of the integral.

This simple but crucial observation will be needed below for the demonstration that Pohlmeyer-invariants map the boundary state of a bare D-brane to that describing a brane with a nonabelian gauge field turned on.

7.2.6 Super-Pohlmeyer and boundary states

We now have all ingredients in place to apply the super-Pohlmeyer invariant to the boundary state of a bare D9 brane. A brief review of the idea of boundary states adapted to the present context is given in [30], but in fact only two simple relations are needed for the following:

If $|\mathrm{D9}\rangle$ is the boundary state of the space-filling BPS D9 brane, then (due to equation (2.26) in [28] and section 2.3.1 in [30]) we have

$$\mathcal{P}^\mu(\sigma)\,|\mathrm{D9}\rangle = \sqrt{\frac{T}{2}}X'^\mu(\sigma)\,|\mathrm{D9}\rangle \tag{7.111}$$

and

$$\Gamma^\mu(\sigma)\,|\mathrm{D9}\rangle = \mathcal{E}^{\dagger\mu}(\sigma)\,|\mathrm{D9}\rangle\ . \tag{7.112}$$

Using the results of §7.2.5 (p.156) such a replacement extends to the full Wilson line made up from these objects:

Consider the extension (7.103) of the restricted super-Pohlmeyer invariant with $A_+ \neq 0$ and furthermore only mutually commuting spatial components of A nonvanishing. In this case the fermionic terms in the integrand have trivial commutators so that the integrand as a whole satisfies condition (7.109). Therefore, according to the result of §7.2.5 (p.156), we can move all appearances of $\mathcal{P}^\mu + \frac{1}{4}(F_A)_{\mu\nu}\Gamma^\mu\Gamma^\nu$ to the boundary state $|\mathrm{D9}\rangle$ on the right, change it there to $\sqrt{\frac{T}{2}}X'^\mu + \frac{1}{4}(F_A)_{\mu\nu}\mathcal{E}^{\dagger\mu}\mathcal{E}^{\dagger\nu}$ and then move this back to the original position (noting that still $[X'(\sigma),\mathcal{P}(\kappa)] \propto \delta'(\sigma-\kappa)$). This way we have

$$\begin{aligned}
&\mathrm{Tr}\,\mathrm{P}\exp\left(\int_0^{2\pi} d\sigma\ \left(iA_\mu\mathcal{P}^\mu + \frac{1}{4}(F_A)_{\mu\nu}\Gamma^\mu\Gamma^\nu\right)\right)|\mathrm{D9}\rangle \\
&= \mathrm{Tr}\,\mathrm{P}\exp\left(\int_0^{2\pi} d\sigma\ \left(-i\sqrt{\frac{T}{2}}A_\mu X'^\mu + \frac{1}{4}(F_A)_{\mu\nu}\mathcal{E}^{\dagger\mu}\mathcal{E}^{\dagger\nu}\right)\right)|\mathrm{D9}\rangle\ .
\end{aligned} \tag{7.113}$$

If we allowed ourself to regulate all the generalized Wilson lines considered here by a point-splitting method as in [152], i.e. by taking care that no local fields in the Wilson line ever come closer than some samll distance σ, then the above step becomes a triviality. Indeed, the result of [152] together with those of [154, 155, 30] shows that this is a viable approach, because the condition for the ϵ-regularized Wilson line to be still an invariant is the same as that of the non-regularized Wilson line to be free of divergences and hence well defined.

It will be convenient for our purposes to rescale A as

$$A \mapsto -\sqrt{\frac{2}{T}}A\,, \tag{7.114}$$

so that this becomes

$$\cdots = \operatorname{Tr} \mathrm{P} \exp\left(\int_0^{2\pi} d\sigma\ \left(iA_\mu X'^\mu + \frac{1}{2T}(F_A)_{\mu\nu}\mathcal{E}^{\dagger\mu}\mathcal{E}^{\dagger\nu}\right)\right)|\mathrm{D9}\rangle\ . \tag{7.115}$$

This is finally our main result, because this is precisely the boundary state of a nonabelian gauge field as considered in equation (3.14) of [30], which is a generalization of the abelian case studied in [154, 155]. The same form of the boundary state is obtained from equations (3.7), (3.8) in [152] when in the expression given there the integral over the Grassmann variables is performed (following the computation described on pp. 236-237 of [157]).

The boundary state (7.115) has two important properties:

1. **7.2.6.1 Super-Ishibashi property of the boundary state.** The defining property of boundary states is that they are annihilated by the generators $\mathcal{L}_K$ of σ-reparameterization as well as, in the superstring case, by their square root $\mathbf{d}_K$, which is a deformed exterior derivative on loop space. $\mathcal{L}_K$ is a linear combination of left- and right-moving bosonic super-Virasoro generators, while $\mathbf{d}_K$ is a combination of fermionic super-Virasoro generators, as discussed in [30].

   It is noteworthy that the state (7.115) indeed satisfies the Ishibashi conditions. Naively this must be the case, because this state is obtained from the bare $|\mathrm{D9}\rangle$, which does satisfy it by definition, by acting on it with a super-Pohlmeyer operator, that commutes with all constraints and hence leaves the Ishibashi property of $|\mathrm{D9}\rangle$ intact. But above we mentioned that the restricted form (7.100) of the quantized Pohlmeyer invariants that this state comes from has potential quantum anomalies which would spoil this invariance. These are due to the non-Grassmann property of the quantized fermions Γ. However, after application to the bare $|\mathrm{D9}\rangle$ which gives (7.115), the left- and right-moving fermions are replaced by their polar combination $\mathcal{E}^\dagger$, and these again enjoy the Grassmann property (they are nothing but differential forms on loop space). For this reason the final result can again enjoy the Ishibashi property, which means nothing but super-reparameterization invariance with respect to σ.

   Proof:

   The invariance under reparameterizations induced by $\mathcal{L}_K$ is manifest, analogous in all the cases considered here before, since (7.115) is the generalized Wilson line over an object of unit reparameterization weight.

   The only nontrivial part that hence needs to be checked is the commutation with $\mathbf{d}_K$ and here we only need to know that $[\mathbf{d}_K, X^\mu(\sigma)] = \mathcal{E}^{\dagger\mu}(\sigma)$.

   Applying this to (7.115) we get, in the same manner as in the similar computations before, from $[\mathbf{d}_K, iA_\mu X'^\mu] = i(\partial_\mu A_\nu - \partial_\nu A_\mu)\mathcal{E}^{\dagger\mu}X'^\nu + \left(iA_\mu\mathcal{E}^{\dagger\mu}\right)'$ coalesced terms $-\left[A_\mu, A_\nu\right]\mathcal{E}^{\dagger\mu}X'^\nu$ and $\frac{i}{2T}\left[A_\kappa, (F_A)_{\mu\nu}\right]\mathcal{E}^{\dagger\kappa}\mathcal{E}^{\dagger\mu}\mathcal{E}^{\dagger\nu}$ at the integration boundaries.

   These combine with the terms $\left[\mathbf{d}_K, \frac{1}{2T}(F_A)_{\mu\nu}\mathcal{E}^{\dagger\mu}\mathcal{E}^{\dagger\nu}\right] = \frac{1}{2T}(\partial_{[\kappa}(F_A)_{\mu\nu]}\mathcal{E}^{\dagger\kappa}\mathcal{E}^{\dagger\mu}\mathcal{E}^{\dagger\nu}) +$ $-i(F_A)_{\mu\nu}\mathcal{E}^{\dagger\mu}X'^\nu$ to $\left(i(\partial_\mu A_\nu - \partial_\nu A_\mu) - [A_\mu, A_\nu] - i(F_A)_{\mu\nu}\right)\mathcal{E}^{\dagger\mu}X'^\nu = 0$ and

$\frac{1}{2T}\left(\partial_{[\kappa}(F_A)_{\mu\nu} + i\left[A_{[\kappa}, (F_A)_{\mu\nu]}\right]\right)\mathcal{E}^{\dagger\kappa}\mathcal{E}^{\dagger\mu}\mathcal{E}^{\dagger\nu} = 0$. Hence all terms vanish and $\mathbf{d}_K$ commutes with (7.115). □

2. **7.2.6.2 Nonlinear gauge invariance of the boundary state.** A generic state constructed from gluon vertices for nonabelian A will generically not be invariant under a target space gauge transformation $A \to UAU^\dagger + U(dU^\dagger)$. The generalized Wilson line in (7.115) however does have this invariance - at least at the classical level. This follows from the general invariance properties of Wilson lines (for details see appendix B of [30]) and depends crucially on the appearance of the gauge covariant field strength F_A in (7.115).

8. Local Connections on Loop Space from Worldsheet Deformations

The following is taken from the preprint [30]. It leads over from the SCFT deformation theory in part II to the 2-bundle theory in part III.

8.1 SCFT deformations in loop space formalism

This introductory section discusses aspects of loop space formalism and deformation theory that will be applied in §8.2 (p.168) to the description of nonabelian 2-form background fields.

8.1.1 SCFT deformations and backgrounds using Morse theory technique

The reasoning by which we intend to derive the worldsheet theory for superstrings in nonabelian 2-form backgrounds involves an interplay of deformation theory of superconformal field theories for closed strings, as described in [27], as well as the generalization to boundary state deformations, which are disucssed further below in §8.1.2 (p.165). The deformation method we use consists of adding deformation terms to the super Virasoro generators and in this respect is in the tradition of similiar approaches as for instance described in [115, 116, 117, 118, 119] (as opposed to, say, deformations of the CFT correlators). What is new here is the systematic use of similarity transformations on a certain combination of the supercharges, as explained below.

In this section the SCFT deformation technique for the closed string is briefly reviewed in a manner which should alleviate the change of perspective from the string's Fock space to loop space.

Consider some realization of the superconformal generators $L_n, \bar{L}_n, G_r, \bar{G}_r$ (we follow the standard notation of [129]) of the type II superstring. We are looking for consistent deformations of these operators to operators $L_n^\Phi, \bar{L}_n^\Phi, G_r^\Phi, \bar{G}_r^\Phi$ (Φ indicates some unspecified background field consiguration which is associated with the deformation) which still satisfy the superconformal algebra and so that the generator of spatial worldsheet reparametrizations remains invariant:

$$L_n^\Phi - \bar{L}_{-n}^\Phi \stackrel{!}{=} L_n - \bar{L}_{-n}\,. \tag{8.1}$$

This condition follows from a canonical analysis of the worldsheet action, which is nothing but 1+1 dimensional supergravity coupled to various matter fields. As for all gravitational theories, their ADM constraints break up into spatial diffeomorphism constraints as well as the Hamiltonian constraint, which alone encodes the dynamics.

The condition (8.1) can also be understood in terms of boundary state formalism, which is briefly reviewed in §8.4 (p.178). As discussed below, the operator $\mathcal{B}$ related to a nontrivial bounday state $|\mathcal{B}\rangle$ can be interpreted as inducing a deformation $G_r^\Phi \equiv \mathcal{B}^{-1} G_r \mathcal{B}$, etc. and the condition (8.1) is then equivalent to (8.97).

In any case, we are looking for isomorphisms of the superconformal algebra which satisfy (8.1):

To that end, let d_r and $d_r^\dagger$ be the modes of the polar combinations of the left- and right-moving supercurrents

$$\begin{aligned} d_r &\equiv G_r + i\bar{G}_{-r} \\ d_r^\dagger \equiv (d_r)^\dagger &= G_r - i\bar{G}_{-r}\,. \end{aligned} \tag{8.2}$$

These are the 'square roots' of the reparametrization generator

$$\mathcal{L}_n \equiv -i\left(L_n - \bar{L}_{-n}\right)\,, \tag{8.3}$$

i.e.

$$\{d_r, d_s\} = \{d_r^\dagger, d_s^\dagger\} = 2i\mathcal{L}_{r+s}\,. \tag{8.4}$$

Under a deformation the right hand side of this equation must stay invariant (8.1) so that

$$\begin{aligned} d_r^\Phi &\equiv d_r + \Delta_\Phi d_r \\ d_r^{\dagger\Phi} &\equiv d_r^\dagger + (\Delta_\Phi d_r)^\dagger \end{aligned} \tag{8.5}$$

implies that the shift $\Delta_\Phi d_r$ of d_r has to satisfy

$$\{d_r, \Delta_\Phi d_s\} + \{d_s, \Delta_\Phi d_r\} + \{\Delta_\Phi d_r, \Delta_\Phi d_s\} = 0\,. \tag{8.6}$$

One large class of solutions of this equation is

$$\Delta_\Phi d_r = A^{-1}\left[d_r, A\right]\,, \qquad \text{for } [\mathcal{L}_n, A] = 0 \;\forall\, n\,, \tag{8.7}$$

where A is any even graded operator that is spatially reparametrization invariant, i.e. which commutes with (8.3).

When this is rewritten as

$$\begin{aligned} d_r^\Phi &= A^{-1}\circ d_r \circ A \\ d_r^{\dagger\Phi} &= A^\dagger \circ d_r^\dagger \circ A^{\dagger -1} \end{aligned} \tag{8.8}$$

one sees explicitly that the formal structure involved here is a direct generalization of that used in [3] in the study of the relation of deformed generators in supersymmetric quantum *mechanics* to Morse theory. Here we are concerned with the direct generalization of this mechanism from $1+0$ to $1+1$ dimensional supersymmetric field theory.

In $1+0$ dimensional SQFT (i.e. supersymmetric quantum mechanics) relation (8.8) is sufficient for the deformation to be truly an isomorphism of the algebra of generators. In 1+1 dimensions, on the superstring's worldsheet, there is however one further necessary condition for this to be the case. Namely the (modes of the) new worldsheet Hamiltonian constraint $H_n = L_n + \bar{L}_-n$ must clearly be defined as

$$H_n^\Phi \equiv \frac{1}{2}\left\{d_r^\Phi, d_{n-r}^{\dagger\Phi}\right\} - \delta_{n,0}\frac{c}{12}\left(4r^2 - 1\right) \tag{8.9}$$

and (8.6) alone does not guarantee that this is *unique* for all $r \neq n/2$. *If* it is, however, then the Jacobi identity already implies that

$$
\begin{aligned}
G_r^\Phi &\equiv \frac{1}{2}\left(d_r^\Phi + d_r^{\dagger\Phi}\right) \\
L_n^\Phi &\equiv \frac{1}{4}\left(\left\{d_r^\Phi, d_{n-r}^{\dagger\Phi}\right\} + \left\{d_r^\Phi, d_{n-r}^\Phi\right\}\right) - \delta_{r,n/2}\frac{c}{24}\left(4r^2-1\right) \\
\bar{G}_r^\Phi &\equiv -\frac{i}{2}\left(d_{-r}^\Phi - d_{-r}^{\dagger\Phi}\right) \\
L_n^\Phi &\equiv \frac{1}{2}\left(\left\{d_{-r}^\Phi, d_{r-n}^{\dagger\Phi}\right\} - \left\{d_{-r}^\Phi, d_{r-n}^\Phi\right\}\right)\,, \qquad \forall\, r \neq n/2
\end{aligned}
\tag{8.10}
$$

generate two mutually commuting copies of the super Virasoro algebra.

In order to see this first note that the two copies of the unperturbed Virasoro algebra in terms of the 'polar' generators $d_r, d_r^\dagger, i\mathcal{L}_m, H_m$ read

$$
\begin{aligned}
&\{d_r, d_s\} = 2\,i\mathcal{L}_{r+s} = \left\{d_r^\dagger, d_s^\dagger\right\} \\
&[i\mathcal{L}_m, d_r] = \frac{m-2r}{2}d_{m+r} \\
&\left[i\mathcal{L}_m, d_r^\dagger\right] = \frac{m-2r}{2}d_{m+r}^\dagger \\
&[i\mathcal{L}_m, i\mathcal{L}_n] = (m-n)i\mathcal{L}_{m+n} \\
&[i\mathcal{L}_m, H_n] = (m-n)i\mathcal{H}_{m+n} + \frac{c}{6}(m^3-m)\delta_{m,-n} \\
&[H_m, d_r] = \frac{m-2r}{2}d_{m+r}^\dagger \\
&\left[H_m, d_r^\dagger\right] = \frac{m-2r}{2}d_{m+r} \\
&[H_m, H_n] = (m-n)\,i\mathcal{L}_{m+n}\,.
\end{aligned}
\tag{8.11}
$$

Now check that these relations are obeyed also by the deformed generators d_r^Φ, $d_r^{\dagger\Phi}$, $i\mathcal{L}_m$, H_m^Φ using the two conditions (8.8) and (8.9):

First of all the relations

$$
\begin{aligned}
\left[i\mathcal{L}_m, d_r^\Phi\right] &= \frac{m-2r}{2}d_{m+r}^\Phi \\
\left[i\mathcal{L}_m, d_r^{\dagger\Phi}\right] &= \frac{m-2r}{2}d_{m+r}^{\dagger\Phi}
\end{aligned}
\tag{8.12}
$$

follow simply from (8.8) and the original bracket $[L_m, G_r] = \frac{m-2r}{2}G_{m+r}$ and immediately imply

$$
[i\mathcal{L}_m, i\mathcal{L}_n] = (m-n)i\mathcal{L}_{m+n}
\tag{8.13}
$$

(note that here the anomaly of the left-moving sector cancels that of the right-moving one).

Furthermore

$$\begin{aligned}
\left[i\mathcal{L}_m, H_n^\Phi\right] &= \left[i\mathcal{L}_m, \frac{1}{2}\left\{d_r^\Phi, d_{n-r}^{\dagger\Phi}\right\}\right] \\
&\overset{(8.12)}{=} \frac{m-2r}{4}\left\{d_{m+r}^\Phi, d_{n-r}^{\dagger\Phi}\right\} + \frac{m-2(n-r)}{4}\left\{d_r^\Phi, d_{m+n-r}^{\dagger\Phi}\right\} \\
&\overset{(8.9)}{=} (m-n)H_{m+n}^\Phi + \delta_{m,-n}\frac{c}{6}\left(\frac{m-2r}{4}(4(m+r)^2-1) + \frac{m-2(n-r)}{4}(4r^2-1)\right) \\
&= (m-n)H_{m+n}^\Phi + \delta_{m,-n}\frac{c}{6}\left(m^3-m\right) . \qquad (8.14)
\end{aligned}$$

(Here the anomalies from both sectors add.)

The commutator of the Hamiltonian with the supercurrents is obtained for instance by first writing:

$$\begin{aligned}
\left[H_m^\Phi, d_r^\Phi\right] &= \frac{1}{2}\left[\left\{d_r^\Phi, d_{m-r}^{\dagger\Phi}\right\}, d_r^\Phi\right] \\
&= -\frac{1}{2}\left[\left\{d_r^\Phi, d_r^\Phi\right\}, d_{m-r}^{\dagger\Phi}\right] - \frac{1}{2}\left[\left\{d_r^\Phi, d_{m-r}^{\dagger\Phi}\right\}, d_r^\Phi\right] \\
&= -\left[i\mathcal{L}_{2r}, d_{m-r}^{\dagger\Phi}\right] - \left[H_m^\Phi, d_r^\Phi\right] \\
&= (m-2r)d_{m+r}^{\dagger\Phi} - \left[H_m^\Phi, d_r^\Phi\right] , \qquad (8.15)
\end{aligned}$$

from which it follows that

$$\left[H_m^\Phi, d_r^\Phi\right] = \frac{(m-2r)}{2}d_{m+r}^{\dagger\Phi} \qquad (8.16)$$

and similarly

$$\left[H_m^\Phi, d_r^{\dagger\Phi}\right] = \frac{(m-2r)}{2}d_{m+r}^\Phi\,. \qquad (8.17)$$

This can finally be used to obtain

$$\left[H_m^\Phi, H_n^\Phi\right] = (m-n)i\mathcal{L}_{m+n}\,. \qquad (8.18)$$

In summary this shows that every operator A which

1. commutes with $i\mathcal{L}_m$
2. is such that $\left\{A^{-1}d_rA, A^\dagger d_{n-r}^\dagger A^{\dagger-1}\right\} - \delta_{n,0}\frac{c}{12}(4r^2-1)$ is *independent* of r

defines a consistent deformation of the super Virasoro generators and hence a string background which satisfies the classical equations of motion of string field theory.

In [27] it was shown how at least all massless NS and NS-NS backgrounds can be obtained by deformations A of the form $A = e^{\mathbf{W}}$, where $\mathbf{W}$ is related to the vertex operator of the respective background field. For instance a Kalb-Ramond B-field background is induced by setting

$$\mathbf{W}^{(B)} = \frac{1}{2}\int d\sigma\ \left(\frac{1}{T}dA + B\right)_{\mu\nu}\mathcal{E}^{\dagger\mu}\mathcal{E}^{\dagger\nu}\,, \qquad (8.19)$$

where $\mathcal{E}^\dagger$ are operators of exterior multiplication with differential forms on loop space, to be discussed in more detail below in §8.1.3.1 (p.166), and we have included the well known contribution of the 1-form gauge field A.

Moreover, it was demonstrated in [113] that the structure (8.8) of the SCFT deformations allows to handle superstring evolution in nontrivial backgrounds as generalized Dirac-Kähler evolution in loop space.

In the special case where A is *unitary* the similarity transformations (8.8) of d and $d^\dagger$ and hence of all other elements of the super-Virasoro algebra are identical and the deformation is nothing but a unitary transformation. It was discussed in [27] that gauge transformations of the background fields, such as reparameterizations or gauge shifts of the Kalb-Ramond field, are described by such unitary transformation.

In particular, an *abelian* gauge field background was shown to be induced by the Wilson line

$$\mathbf{W}^{(A)} = i \oint d\sigma\; A_\mu(X(\sigma))\, X'^\mu(\sigma) \tag{8.20}$$

of the gauge field along the closed string.

While the above considerations apply to closed superstrings, in this paper we shall be concerned with open superstrings, since these carry the Chan-Paton factors that will transform under the nonabelian group that we are concerned with in the context on nonablian 2-form background fields.

It turns out that the above method for obtaining closed string backgrounds by deformations of the differential geometry of loop space nicely generalizes to open strings when boundary state formalism is used. This is the content of the next section.

8.1.2 Boundary state deformations from unitary loop space deformations

The tree-level diagram of an open string attached to a D-brane is a disk attached to that brane with a certain boundary condition on the disk characterizing the presence of the D-brane. In what is essentially a generalization of the method of image charges in electrostatics this can be equivalently described by the original disk "attached" to an auxiliary disc, so that a sphere is formed, and with the auxiliary disk describing incoming closed strings in just such a way, that the correct boundary condition is reproduced.

Some details behind this heuristic picture are recalled in §8.4 (p.178). For our purposes it suffices to note that a deformation (8.8) of the superconformal generators for closed strings with A a *unitary* operator (as for instance given by (8.20)) is of course equivalent to a corresponding unitary transformation of the closed string states. But this means that the boundary state formalism implies that open string dynamics in a given background described by a unitary deformation operator A on loop space is described by a boundary state $A^\dagger\,|D9\rangle$, where $|D9\rangle$ is the boundary state of a bare space-filling brane, which again, as discussed below in (8.30), is nothing but the constant 0-form on loop space.

In this way boundary state formalism rather nicely generalizes the loop space formalism used here from closed to open strings.

In a completely different context, the above general picture has in fact been verified for abelian gauge fields in [154, 155]. There it is shown that acting with (8.20) and the unitary

part[9] of (8.19) for $B = 0$ on $|D9\rangle$, one obtains the correct boundary state deformation operator

$$\exp\left(\mathbf{W}^{(A)(B=\frac{1}{T}dA)}\right)\exp\left(\int\limits_0^{2\pi}\left(iA_\mu X'^\mu + \frac{1}{2T}(dA)_{\mu\nu}\mathcal{E}^{\dagger\mu}\mathcal{E}^{\dagger\nu}\right)\right) \tag{8.21}$$

which describes open strings on a D9 brane with the given gauge field turned on.

Here, we want to show how this construction directly generalizes to deformations describing nonabelian 1- and 2-form backgrounds. It turns out that the loop space perspective together with boundary state formalism allows to identify the relation between the nonablian 2-form background and the corresponding connection on loop space, which again allows to get insight into the gauge invariances of gauge theories with nonabelian 2-forms.

One simple observation of the abelian theory proves to be crucial for the non-abelian generalization: Since (8.21) *commutes* with $\mathbf{d}_K$ the loop space connection it induces (following the reasoning to be described in §8.1.3.2 (p.168)) *vanishes*. This makes good sense, since the closed string does not feel the background A field.

But the generalization of a *vanishing* loop space connection to something less trivial but still trivial enough so that it can describe something which does not couple to the closed string is a *flat* loop space connection. Flatness in loop space means that every closed curve in loop space, which is a torus worldsheet (for the space of oriented loops) in target space, is assigned surface holonomy $g = 1$, the identity element. This means that only open worldsheets with boundary can feel the presence of a flat loopspace connection, just as it should be.

From this heuristic picture we expect that abelian but flat loop space connections play a special role. Indeed, we shall find in §8.2.3 (p.171) that only these are apparently well behaved enough to avoid a couple of well known problems.

The next section first demonstrates that the meaning of the above constructions become rather transparent when the superconformal generators are identified as deformed deRham operators on loop space.

8.1.3 Superconformal generators as deformed deRham operators on loop space.

Details of the representation of the super Virasoro generators on loop space have been given in [27] and we here follow the notation introduced there.

8.1.3.1 Differential geometry on loop space. Again, the loop space formulation can nicely be motivated from boundary state formalism:

The boundary state $|\mathrm{b}\rangle$ describing the space-filling brane in Minkowski space is, according to (8.96), given by the constraints

$$\begin{aligned} \left(\alpha_n^\mu + \bar{\alpha}_{-n}^\mu\right)|\mathrm{b}\rangle &= 0\,, \qquad \forall\, n,\mu \\ \left(\psi_r^\mu - i\bar{\psi}_{-r}^\mu\right)|\mathrm{b}\rangle &= 0\,, \qquad \forall\, r,\mu \end{aligned} \tag{8.22}$$

[9] When acting on $|D9\rangle$ the non-unitary part of (8.19) is projected out automatically.

(in the open string R sector).

We can think of the super-Virasoro constraints as a Dirac-Kähler system on the exterior bundle over loop space $\mathcal{L}(M)$ with coordinates

$$X^{(\mu,\sigma)} = \frac{1}{\sqrt{2\pi}} X_0^\mu + \frac{i}{\sqrt{4\pi T}} \sum_{n\neq 0} \frac{1}{n} \left(\alpha_n^\mu - \tilde{\alpha}_{-n}^\mu\right) e^{in\sigma} , \tag{8.23}$$

holonomic vector fields

$$\frac{\delta}{\delta X^\mu(\sigma)} \equiv \partial_{(\mu,\sigma)} = i\sqrt{\frac{T}{4\pi}} \sum_{n=-\infty}^{\infty} \eta_{\mu\nu} \left(\alpha_n^\nu + \tilde{\alpha}_{-n}^\nu\right) e^{in\sigma} \tag{8.24}$$

differential form creators

$$\begin{aligned} \mathcal{E}^{\dagger(\mu,\sigma)} &= \frac{1}{2}\left(\psi_+^\mu(\sigma) + i\psi_-^\mu(\sigma)\right) \\ &= \frac{1}{\sqrt{2\pi}} \sum_r \left(\bar{\psi}_{-r} + i\psi_r\right) e^{ir\sigma} \end{aligned} \tag{8.25}$$

and annihilators

$$\begin{aligned} \mathcal{E}^{(\mu,\sigma)} &= \frac{1}{2}\left(\psi_+^\mu(\sigma) - i\psi_-^\mu(\sigma)\right) \\ &= \frac{1}{\sqrt{2\pi}} \sum_r \left(\bar{\psi}_{-r} - i\psi_r\right) e^{ir\sigma} . \end{aligned} \tag{8.26}$$

In the polar form (8.2) the fermionic super Virasoro constraints are identified with the modes of the exterior derivative on loop space

$$\mathbf{d}_K = \int_0^{2\pi} d\sigma \, \left(\mathcal{E}^{\dagger\mu}\partial_\mu(\sigma) + iTX'^\mu \mathcal{E}_\mu(\sigma)\right) , \tag{8.27}$$

deformed by the reparametrization Killing vector

$$K^{(\mu,\sigma)} \equiv X'^\mu(\sigma) , \tag{8.28}$$

where $T = \frac{1}{2\pi\alpha'}$ is the string tension. The Fourier modes of this operator are the polar operators of (8.2)

$$d_r \propto \oint d\sigma \, e^{-ir\sigma} \mathbf{d}_K(\sigma) . \tag{8.29}$$

Using this formulation of the super-Virasoro constraints it would seem natural to represent them on a Hilbert space whose 'vacuum' state $|\mathrm{vac}\rangle$ is the *constant 0-form* on loop space, i.e.

$$\partial_{(\mu,\sigma)} |\mathrm{vac}\rangle = 0 = \mathcal{E}_{(\mu,\sigma)} |\mathrm{vac}\rangle \qquad \forall \, \mu, \sigma . \tag{8.30}$$

While this is not the usual $\mathrm{SL}(2,\mathbf{c})$ invariant vacuum of the closed string, it is precisely the boundary state (8.22)

$$|\mathrm{vac}\rangle = |\mathrm{b}\rangle \tag{8.31}$$

describing the D9 brane.

For the open string NS sector the last relation of (8.22) changes the sign

$$\left(\psi^\mu_r + i\bar{\psi}^\mu_{-r}\right)\left|\mathrm{b}'\right\rangle = 0\,, \qquad \forall\, r,\mu \;\; \text{NS sector} \tag{8.32}$$

and now implies that the vacuum is, from the loop space perspective, the formal *volume form* instead of the constant 0-form, i.e. that form annihilated by all differential form multiplication operators:

$$\partial_{(\mu,\sigma)}\left|\mathrm{b}'\right\rangle = 0 = \mathcal{E}^{\dagger(\mu,\sigma)}\left|\mathrm{b}'\right\rangle \qquad \forall\, \mu,\sigma\,. \tag{8.33}$$

In finite dimensional flat manifolds of course both are related simply by *Hodge duality*:

$$\left|\mathrm{b}'\right\rangle = \star\left|\mathrm{b}\right\rangle\,. \tag{8.34}$$

So Hodge duality on loop space translates to the NS $\leftrightarrow$ R transition on the open string sectors.

8.1.3.2 Connections on loop space. It is now straightforward to identify the relation between background fields induced by deformations (8.8) and connections on loop space. A glance at (8.27) shows that we have to interpret the term of differential form grade +1 in the polar supersymmetry generator as $\mathcal{E}^\dagger\hat{\nabla}^{(\Phi)}_\mu$, where $\hat{\nabla}^{\Phi}_\mu$ is a loop space connection (covariant derivative) induced by the target space background field Φ.

Indeed, as was shown in [27], one finds for instance that a gravitational background $G_{\mu\nu}$ leads to $\hat{\nabla}^{(G)}$ which is just the Levi-Civita connection on loop space with respect to the metric induced from target space. Furthermore, an *abelian* 2-form field background is associated with a deformation operator

$$\mathbf{W}^{(B)} = \oint d\sigma\; B_{\mu\nu}\mathcal{E}^{\dagger\mu}\mathcal{E}^{\dagger\nu} \tag{8.35}$$

and leads to a connection

$$\hat{\nabla}^{(G)(B)}_\mu = \hat{\nabla}^{(G)}_\mu - iTB_{\mu\nu}X'^\nu\,, \tag{8.36}$$

just as expected for a string each of whose points carries $U(1)$ charge under B proportional to the length element $X'\,d\sigma$.

8.2 BSCFT deformation for nonabelian 2-form fields

The above mentioned construction can now be used to examine deformations that involve nonabelian 2-forms:

8.2.1 Nonabelian Lie-algebra valued forms on loop space

When gauge connections on loop space take values in nonabelian algebras deformation operators such as $\exp\left(\mathbf{W}^{(A)}\right)$ (8.20) and $\exp\left(\mathbf{W}^{(B)}\right)$ (8.35) obviously have to be replaced by path ordered exponentiated integrals. The elementary properties of loop space differential

forms involving such path ordered integrals are easily derived, and were for instance given in [158, 53].

So consider a differential $p+1$ form ω on *target space*. It lifts to a $p+1$-form Ω on loop space given by

$$\Omega \equiv \frac{1}{(p+1)!} \int\limits_{S^1} \omega_{\mu_1\cdots\mu_{p+1}}(X)\, \mathcal{E}^{\dagger\mu_1} \dots \mathcal{E}^{\dagger\mu_{p+1}} \,. \tag{8.37}$$

Let $\hat{K} = X'^{(\mu,\sigma)}\mathcal{E}_{(\mu,\sigma)}$ be the operator of interior multiplication with the reparametrization Killing vector K (8.28) on loop space. The above $p+1$-form is sent to a p-form $\oint(\omega)$ on loop space by contracting with this Killing vector (brackets will always denote the graded commutator):

$$\begin{aligned} \oint(\omega) &\equiv \left[\hat{K}, \Omega\right] \\ &= \frac{1}{p!} \int\limits_{S^1} d\sigma\; \omega_{\mu_1\cdots\mu_{p+1}} X'^{\mu_1} \mathcal{E}^{\dagger\mu_2} \dots \mathcal{E}^{\dagger\mu_{p+1}} \,. \end{aligned} \tag{8.38}$$

The anticommutator of the loop space exterior derivative $\mathbf{d}$ with $\hat{K}$ is just the reparametrization Killing Lie derivative

$$\left[\mathbf{d}, \hat{K}\right] = i\mathcal{L}_K \tag{8.39}$$

which commutes with 0-modes of fields of definite reparametrization weight, e.g.

$$[\mathcal{L}, \Omega] = 0 \,. \tag{8.40}$$

It follows that

$$\left[\mathbf{d}, \left[\hat{K}, \Omega\right]\right] = [\mathcal{L}, \Omega] - \left[\hat{K}, [\mathbf{d}, \Omega]\right] \tag{8.41}$$

which implies that

$$\left[\mathbf{d}, \oint(\omega)\right] = \oint(-d\omega) \,. \tag{8.42}$$

The generalization to multiple path-ordered integrals

$$\oint(\omega_1, \cdots, \omega_n) \equiv \int\limits_{0<\sigma_{i-1}<\sigma_i<\sigma_{i+1}<\pi} d^n\sigma \;\left[\hat{K}, \omega_1\right](\sigma_1) \cdots \left[\hat{K}, \omega_n\right](\sigma_n) \tag{8.43}$$

is

$$\begin{aligned} &\left[\mathbf{d}, \oint(\omega_1, \cdots, \omega_n)\right] = \\ &= -\sum_k (-1)^{\sum\limits_{i<k} p_i} \left(\oint(\omega_1, \cdots, d\omega_k, \cdots, \omega_n) + \oint(\omega_1, \cdots, \omega_{k-1} \wedge \omega_k, \cdots, \omega_n) \right) . \end{aligned} \tag{8.44}$$

This is proposition 1.6 in [158].

In the light of (8.8) we are furthermore interested in expressions of the form $U_A(2\pi,0)\circ \mathbf{d}_K \circ U_A(0,2\pi)$ where U_A is the holonomy of A.

Using

$$
\begin{aligned}
[\mathbf{d}, U_A(0,2\pi)] &= \left[\mathbf{d}, \sum_{n=0}^{\infty} \oint \underbrace{(iA,\cdots,iA)}_{n \text{ times}}\right] \\
&= -\sum_{n=0}^{\infty}\sum_{k} \oint (iA,\cdots,iA,iF_A,iA,\cdots iA)_{n \text{ occurences of } iA,\, F_A \text{ at } k} \\
&= \int_0^{2\pi} d\sigma\, U_A(0,\sigma)\left[iF_A,\hat{K}\right](\sigma)\, U(\sigma,2\pi) \qquad (8.45)
\end{aligned}
$$

where

$$
\begin{aligned}
F_A &= -i(d+iA)^2 \\
&= dA + iA\wedge A \qquad (8.46)
\end{aligned}
$$

is the field strength of A (which is taken to be hermitean), one finds

$$
U_A(2\pi,0)\circ \mathbf{d}\circ U_A(0,2\pi) = \mathbf{d} + \int_0^{2\pi} d\sigma\, U_A(2\pi,\sigma)\left[iF_A,\hat{K}\right](\sigma)\, U_A(\sigma,2\pi)\ . \qquad (8.47)
$$

The point that will prove to be crucial in the following discussion is that there is an A-holonomy on *both* sides of the 1-form factor. The operator on the right describes parallel transport with A from 2π to σ, application of $\left[\mathbf{d}_A A, \hat{K}\right]$ at σ and then parallel transport back from σ to 2π. Following [53] the abbreviating notation

$$
\oint_A(\omega) \equiv \int_0^{2\pi} d\sigma\; U_A(2\pi,\sigma)\left[\hat{K},\omega\right] U_A(\sigma,2\pi) \qquad (8.48)
$$

will prove convenient. (But notice that in (8.48) there is also a factor $U_A(\sigma,2\pi)$ on the *right*, which does not appear in [53].) Using this notation (8.47) is rewritten as

$$
U_A(2\pi,0)\circ \mathbf{d}\circ U_A(0,2\pi) = \mathbf{d} - i\oint_A (F_A)\,. \qquad (8.49)
$$

This expression will prove to play a key role in the further development. In order to see why this is the case we now turn to the computation and disucssion of the connection on loop space which is induced by the nonabelian 2-form background.

8.2.2 Nonabelian 2-form field deformation

With the above considerations it is now immediate how to incorporate a nonabelian 2-form in the target space of a boundary superconformal field theory on the worldsheet. The direct

generalization of (8.20) and (8.35) is obviously the deformation operator

$$\exp(\mathbf{W})^{(A)(B)_{\text{nonab}}} = \mathrm{P}\exp\left(\int\limits_0^{2\pi} d\sigma\ \left(iA_\mu X'^\mu + \frac{1}{2}\left(\frac{1}{T}F_A + B\right)_{\mu\nu}\mathcal{E}^{\dagger\mu}\mathcal{E}^{\dagger\nu}\right)\right) \tag{8.50}$$

for non-abelian and hermitean A and B. (P denotes path ordering) Note that this is indeed reparametrization invariant on $\mathcal{L}(M)$ and that there is no trace in (8.50), so that this must act on an appropriate bundle, which is naturally associated with a stack of N branes (*cf.* pp. 3-4 of [159]).

According to §8.1.3.2 (p.168) the loop space connection induced by this deformation operator is given by the term of degree +1 in the deformation of the superconformal generator (8.27). Using (8.47) this is found to be

$$\begin{aligned}\exp(-\mathbf{W})^{(A)(B)_{\text{nonab}}} \circ \mathbf{d}_K \circ \exp(\mathbf{W})^{(A)(B)_{\text{nonab}}} &= \mathbf{d} + iT\oint_A (B) \\ &\quad + (\text{terms of grade} \neq 1)\,,\end{aligned} \tag{8.51}$$

where the notation (8.48) is used.

The second term $iT\oint_A(B)$ is the nonablian 1-form connection on loop space which is induced by the target space 2-form B. Note that the terms involving the A-field strength $d_A A$ coming from the X' term and those coming from the $\mathcal{E}^\dagger\mathcal{E}^\dagger$ term in (8.51) mutually cancel.

The connection (8.51) is essentially that found, by different means in different cntexts, in [53],[160] and [161].

8.2.3 2-Form Gauge Transformations

In a gauge theory with a nonabelian 2-form one expects the usual gauge invariance

$$\begin{aligned} A &\mapsto U\,A\,U^\dagger + U(dU^\dagger) \\ B &\mapsto U\,B\,U^\dagger \end{aligned} \tag{8.52}$$

together with some nonabelian analogue of the infinitesimal shift

$$\begin{aligned} A &\mapsto A + \Lambda \\ B &\mapsto B - d_A\Lambda + \cdots \end{aligned} \tag{8.53}$$

familiar from the abelian theory.

With the above results, it should be possible to derive some properties of the gauge invariances of a nonabelian 2-form theory from loop space reasoning. That's because on loop space (8.51) is an ordinary connection 1-form. The ordinary 1-form gauge transformations of that loop space connection should give rise to something like (8.52) and (8.53) automatically.

Indeed, "global" gauge transformations (i.e. position-independent ones) of (8.51) on loop space give rise to (8.52), while infinitesimal gauge transformations on loop space give rise to (8.53), but with correction terms that only have an interpretation on loop space.

More precisely, let $U(X) = U$ be any *constant* group valued function on (a local patch of) loop space and let $V(X) : M \to G$ be such that $\lim_{\epsilon \to 0} V(X)(X(\epsilon)) = V(X)(X(2\pi - \epsilon)) = U$, then

$$U \left(\oint_A (B) \right) U^\dagger + U(dU^\dagger) \;=\; \oint_{A'} (B') \tag{8.54}$$

with

$$\begin{aligned} A' &= V\, A\, V^\dagger + V(dV^\dagger) \\ B' &= V\, B\, V^\dagger\,, \end{aligned} \tag{8.55}$$

which reproduces (8.52).

If, on the other hand, U is taken to be a *nonconstant* infinitesimal gauge transformation with a 1-form gauge parameter Λ of the form

$$U(X) \;=\; 1 - i \oint_A (\Lambda) \tag{8.56}$$

then

$$U \left(\oint_A (B) \right) U^\dagger + U(dU^\dagger) \;=\; \oint_{A+\Lambda} (B + d_A B) + \cdots\,. \tag{8.57}$$

The first term reproduces (8.53), but there are further terms which do not have analogs on target space.

The reason for this problem can be understood from the boundary deformation operator point of view:

Let

$$\mathrm{P}\exp\left(i \int_0^{2\pi} R \right) \;=\; \lim_{N=1/i\epsilon \to \infty} (1 + i\epsilon R(0)) \cdots (1 + i\epsilon R(\epsilon 2\pi)) \cdots (1 + i\epsilon R(2\pi)) \tag{8.58}$$

be the path ordered integral over some object R. Then a small shift $R \to R + \delta R$ amounts to the "gauge transformation"

$$\mathrm{P}\exp\left(i \int_0^{2\pi} R \right) \to \mathrm{P}\exp\left(i \int_0^{2\pi} R \right) \left(1 + i \oint_R \delta R \right) \tag{8.59}$$

to first order in δR. Notice how, using the definition of $\oint_R$ given in (8.48), the term $\oint_R \delta R$ inserts δR successively at all σ in the preceding Wilson line.

So it would seem that $U = 1 - i \oint_R \delta R$ is the correct unitary operator, to that order, for the associated transformation. But the problem ist that R is in general not purely bosonic,

but contains fermionic contributions. These spoil the ordinary interpretation of the above U as a gauge transformation.

This means that an ordinary notion of gauge transormation is obtained if and only if the fermionic contributions in (8.50) disappear, which is the case when

$$B = -\frac{1}{T} F_A \,. \tag{8.60}$$

Then

$$\exp(\mathbf{W})^{(A)(B=-\frac{1}{T}F_A)} = \mathrm{P}\exp\left(\int_0^{2\pi} d\sigma\; iA_\mu X'^\mu(\sigma)\right) \tag{8.61}$$

is the pure A-Wilson line and and the corresponding gauge covariant exterior derivative on loop space is

$$\mathbf{d}^{(A)(B)} = \mathbf{d} - i\oint_A (F_A)\,, \tag{8.62}$$

as in (8.49). Now the transformation

$$\begin{aligned} A &\mapsto A + \Lambda \\ B &\mapsto B - \frac{1}{T}\left(d\Lambda + iA\wedge\Lambda + i\Lambda\wedge A\right) \end{aligned} \tag{8.63}$$

is correctly, to first order, induced by the loop space gauge transformation

$$U(X) = 1 - i\oint_A (\Lambda)\,. \tag{8.64}$$

The restriction (8.60) was previously found in [92] in the context of categorified lattice gauge theory. The discussion there uses the generalization of the above considerations from the case where A and B take values in the same algebra to that where they take values in a differential crossed module. The above loop space considerations are generalized to this case in §11.5 (p.268). See in particular Prop. (11.10) (p. 275).

8.3 Gerstenhaber Brackets and Hochschild Cohomology

In [53] C. Hofman had presented some considerations which in some respects are closely related to the discussion above, though different. Here we briefly review the key observations of [53] and discuss how they are related to the above. Interestingly, this involves considerations rather similar to but again different from some aspects to be discussed in §13 (p.335). While at the moment this appears like a coincidence, it could be that there is something deeper going on which is not yet fully understood.

In [53] it was noted that equation (8.44) suggests that in differential calculus on pull-back forms on path space *graded multi-derivations* play an important role.

Clearly, if we consider the string of forms $(\omega_1, \ldots, \omega_n)$ as an n-fold associative abstract product of some unspecified sort and if we associate a grade

$$|\omega_i| \equiv p_i = \deg(\omega_i) - 1 \tag{8.65}$$

with each factor then the first term in (8.44) can be thought of as coming from the application of the *unary* derivation ϕ_d of odd grade $|\phi_d| = 1$ defined by[10]

$$\phi_d(\omega_1, \ldots, \omega_n) \equiv \sum_{k=0}^{n-1} (-1)^{\sum_{i=1}^{k} |\phi_d||\omega_i|} (\omega_1, \ldots, \omega_k, -d\omega_{k+1}, \omega_{k+2}, \ldots, \omega_n) \tag{8.66}$$

while the second term can be thought of as due to a *binary* derivation ϕ_M of grade $|\phi_M| = 1$

$$\phi_M(\omega_1, \ldots, \omega_n) \equiv \sum_{k=0}^{n-2} (-1)^{\sum_{i=1}^{k} |\phi_M||\omega_i|} (\omega_1, \ldots, \omega_k, M(\omega_{k+1}, \omega_{k+2}), \ldots, \omega_n) \;, \tag{8.67}$$

where the binary map M is, up to a sign, the (wedge) product operation

$$M(\omega_1, \omega_2) = (-1)^{|\omega_1|} \omega_1 \wedge \omega_2 \,. \tag{8.68}$$

For these two (multi-)derivations it so happens that their grade $|\phi|$ (defined by the signs in the above sums) is related to the grade of their image (defined by (8.65)) simply by

$$|\phi(\omega_1, \ldots, \omega_n)| = |\phi| + \sum_{k=1}^{n} |\omega_k| \,. \tag{8.69}$$

An arbitrary graded multi-derivation has no reason to satisfy this relation. But those that do have the nice property that they form a closed algebra under the graded Lie bracket given by the graded commutator

$$[\phi_1, \phi_2] \equiv \phi_1 \circ \phi_2 - (-1)^{|\phi_1||\phi_2|} \, \phi_2 \circ \phi_1 \,. \tag{8.70}$$

A simple calculation shows that this bracket respects the grade

$$|\,[\phi_1, \phi_2]\,| = |\phi_1| + |\phi_2| \tag{8.71}$$

and that if ϕ_1 is n_1-ary and ϕ_2 is n_2-ary the resulting derivation is $(n_1 + n_2 - 1)$-ary and given by

$$\begin{aligned} &[\phi_1, \phi_2](\omega_1, \ldots, \omega_{n_1+n_2-1}) \\ &= \sum_{r=0}^{n_1-1} (-1)^{\sum_{i=1}^{r} |\phi_2||\omega_i|} \phi_1(\omega_1, \ldots, \omega_r, \phi_2(\omega_{r+1}, \ldots, \omega_{r+n_2}), \omega_{r+n_2+1}, \ldots, \omega_{r+n_1+n_2-1}) \\ &\quad - (-1)^{|\phi_1||\phi_2|} \sum_{r=0}^{n_2-1} (-1)^{\sum_{i=1}^{r} |\phi_1||\omega_i|} \phi_2(\omega_1, \ldots, \omega_r, \phi_1(\omega_{r+1}, \ldots, \omega_{r+n_1}), \omega_{r+n_1+1}, \ldots, \omega_{r+n_1+n_2-1}) \,. \end{aligned} \tag{8.72}$$

[10] We have some (inessential) signs different from those in [53].

As noted in [53], this operation is related to the so-called Gerstenhaber bracket between multilinear maps.

In order to see this, first note that for pull-back forms the grade $|\phi|$ of the multi-derivation which satisfy (8.69) is the sum of the differential form degree they carry and their arity reduced by one, i.e.

$$|\phi| = \deg(\phi) + n_\phi - 1 \tag{8.73}$$

(because they remove n contractions with X' and add one.)

Next it is instructive to first restrict attention to the case where all form degrees of the ωs as well as of the ϕ are *even*, which is the purely 'bosonic' case. In this case the grade $|\omega|$ of a form in (8.65) is *odd* and the grade $|\phi|$ of our multi-derivations is $n-1$.

So in this case (8.72) reduces to the ordinary *Gerstenhaber bracket*

$$\begin{aligned}
[\phi_1, \phi_2] = {} & \phi_1(\phi_2(\ldots), \ldots) - (-1)^{(n_1-1)(n_2-1)}\phi_2(\phi_1(\ldots), \ldots) \\
& + \Big((-1)^{(n_2-1)}\phi_1(\cdot, \phi_2(\ldots), \ldots) - (-1)^{(n_1-1)((n_2-1)+1)}\phi_2(\cdot, \phi_1(\ldots), \ldots)\Big) \\
& + \Big(\phi_1(\cdot, \cdot, \phi_2(\ldots), \ldots) - (-1)^{(n_1-1)(n_2-1)}\phi_2(\cdot, \cdot, \phi_1(\ldots), \ldots)\Big) \\
& + \cdots ,
\end{aligned} \tag{8.74}$$

which reproduces all the familiar algebraic relations between maps: For instance let M be binary, μ unary and α 0-ary (all of even form degree), then (for all their arguments of even form degree, too) the above says that

$$\begin{aligned}
[\mu(\cdot), \alpha] &= \mu(\alpha) \\
[\mu(\cdot), \nu(\cdot)] &= \mu(\nu(\cdot)) - \nu(\mu(\cdot)) \\
[M(\cdot, \cdot), \alpha] &= M(\alpha, \cdot) - M(\cdot, \alpha) \\
[M(\cdot, \cdot), \mu(\cdot)] &= M(\mu(\cdot), \cdot) - M(\cdot, \mu(\cdot)) - \mu(M(\cdot, \cdot)) \\
[M(\cdot, \cdot), M(\cdot, \cdot)] &= 2\,(\, M(M(\cdot, \cdot), \cdot) - M(\cdot, M(\cdot, \cdot))\,)\ .
\end{aligned} \tag{8.75}$$

The expressions here are, respectively,

1. application of a function to its argument
2. commutator of two operators
3. commutator (a measure for the failure of M to define a commutaive product)
4. 'derivator' (a measure for the failure of μ to be a derivation of M)
5. associator (a measure for the failure of M to define an associative product).

Now it is easy to understand the general case where the form degrees may be odd: The above commutators simply become graded commutators and the derivator becomes a

graded derivator in the familiar way. Note that the associator remains intact if for M the product (8.68) is used, so that

$$[M,M] = 0 \tag{8.76}$$

(if the product on any internal degrees of freedom is associative).

With this in hand some interesting things about the exterior derivative on pull-back forms on loop space can be said:

If from now on the letter M is reserved for the special product derivation (8.68) and if d denotes the obvious unary derivation, we can write

$$\mathbf{d} \oint (\omega_1, \ldots, \omega_n) = \oint (d + M)(\omega_1, \ldots, \omega_n) \; . \tag{8.77}$$

The square is

$$\mathbf{d}^2 \oint (\omega_1, \ldots, \omega_n) = \oint \left(\frac{1}{2}\,[d,d] + \frac{1}{2}\,[M,M] + [d,M] \right)(\omega_1, \ldots, \omega_n) \; . \tag{8.78}$$

Using the above insight we find that all three terms vanish by themselves:

$$\begin{aligned}
\frac{1}{2}\,[d,d](\omega_1) &= dd\omega_1 = 0 \\
\frac{1}{2}\,[M,M](\omega_1,\omega_2,\omega_3) &= (-1)^{|\omega_1||\omega_2|} \left((-1)^{|\omega_1|}\omega_1 \wedge \omega_2 \right) \wedge \omega_3 - (-1)^{2|\omega_1|}\omega_1 \wedge \left((-1)^{|\omega_2|}\omega_2 \wedge \omega_3 \right) = 0 \\
[d,M](\omega_1,\omega_2) &= (-1)^{|\omega_1|}\, d(\omega_1 \wedge \omega_2) + (-1)^{|\omega_1|+1}\,(d\omega_1) \wedge \omega_2 + (-1)^{2|\omega_1|}\,\omega_1 \wedge (d\omega_2) \\
&= (-1)^{|\omega_1|} \left(d(\omega_1 \wedge \omega_2) - (d\omega_1) \wedge \omega_2 - (-1)^{\deg(\omega_1)}\,\omega_1 \wedge (d\omega_2) \right) = 0 \, ,
\end{aligned} \tag{8.79}$$

due to the fact that the ordinary exterior derivative is nilpotent and a graded derivation of the ordinary wegde product, which is associative.

In [53] it was argued that one should consider generalizations of the multi-derivation $\phi_d + \phi_M$ using a unary 1-form A and a 0-ary 2-form B multi-derivation (thus using all possible derivations of grade 1), to obtain the derivation $\phi_d + \phi_M + \phi_A + \phi_B$. However, the motivation for this proposal used an argument which was a little shaky (for instance according to that argument there should have really been a term proportional to the field strength of A in equation (15) of [53]). But there is a way to derive this total derivation from an interesting loop space expression:

We know from the tensionful superstring that the natural modification of the exterior derivative on loop space is a polar combination of the worldsheet supercharges, namely the object (6.44) (p. 105)

$$\mathbf{d}_K \equiv \mathbf{d} + iT\,\iota_K \, , \tag{8.80}$$

where ι_K is the operator of inner multiplication with the loop space vector $K^{(\mu,\sigma)} = X'^{\mu}(\sigma)$, which gerenrates rigid reparameterizations, and we have kept the string tension T (a constant) for later discussion of the limit $T \to 0$.

Moreover, from the above we know that the Wilson line of A along the string naturally generalizes to the multi-form

$$W[X](\sigma,\sigma') \equiv \mathrm{P}\exp\left(\int_{\sigma}^{\sigma'} d\sigma \;\left(iA\mu\cdot X'^{\mu} + \frac{1}{2}\left(\frac{1}{T}F_A + B\right)_{\mu\nu} dX^{\mu}\wedge dX^{\nu}\wedge\right)(\sigma)\right)\mathbf{1}$$

$$, \tag{8.81}$$

where $\mathbf{1}$ denotes the constant unit 0-form on loop space. This was our object of interest in §8.2 (p.168).

But now let us try to generalize both this approach as well as the one in [53] and consider modified pull-back forms that have the above generalized Wilson line between each factor:

$$\oint_{(A,B)} (\omega)(\omega_1,\ldots,\omega_n) \equiv \int_{0<\sigma_i<\sigma_{i+1}<1\,\forall\, i} W(0,\sigma_1)\,\iota_K(\omega_1)(\sigma_1)\,W(\sigma_1,\sigma_2)\,\iota_K(\omega_1)(\sigma_1)\cdots W(\sigma_n,1)\;. \tag{8.82}$$

The point of this definition is that the action of the modified exterior derivative $\mathbf{d}_K$ on this object in a certain scaling limit reproduces the action of the multi-derivations proposed in [53] up to an extra term:

If we let B scale as $1/T$ then

$$\mathbf{d}_K \oint_{(A,B)} (\omega_1,\ldots,\omega_n) = \oint_{(A,B)} (d+M+A+B)(\omega_1,\ldots,\omega_n) + \mathcal{O}(1/T)\;, \tag{8.83}$$

where the remaining terms of order $1/T$ have no further contractions with ι_K. Hence there is a scaling limit of *large* string tension $T\to\infty$ with $T\,B$ fixed in which Hofman's multi-derivation are obtained from a proper loop space differential.

Applying $\mathbf{d}_K$ twice yields (*cf.* equation (23) in [53])

$$(d+M+A+B)(d+M+A+B) = \mathcal{H}+\mathcal{F}+\mathcal{N}+\mathcal{K} \tag{8.84}$$

where

$$\begin{aligned}
\mathcal{H} &= [d,B]+[A,B]\\
&= dB + A(B)\\
\mathcal{F}(\omega) &= \left([d,A]+\frac{1}{2}[A,A]+[M,B]\right)(\omega)\\
&= d(A(\omega)) + A(A(\omega)) + M(B,\omega) + (-1)^{|\omega|}M(\omega,B)\\
&= d(A(\omega)) + A(A(\omega)) - B\wedge\omega + \omega\wedge B\\
\mathcal{N}(\omega_1,\omega_2) &= ([d,M]+[A,M])(\omega_1,\omega_2)\\
&= (-1)^{|\omega_1|}\left(d_A(\omega_1\wedge\omega_2) - (d_A\omega_1)\wedge\omega_2 - (-1)^{\deg(\omega_2)}\omega_1\wedge(d_A\omega_2)\right)\\
\mathcal{K}(\omega_1,\omega_2,\omega_3) &= \frac{1}{2}[M,M](\omega_1,\omega_2,\omega_3)\\
&= (-1)^{|\omega_1||\omega_2|}\left((-1)^{|\omega_1|}\omega_1\wedge\omega_2\right)\wedge\omega_3 - (-1)^{2|\omega_1|}\omega_1\wedge\left((-1)^{|\omega_2|}\omega_2\wedge\omega_3\right)\;.
\end{aligned} \tag{8.85}$$

The $\mathcal{H}$ here is the 3-form curvature and $\mathcal{F}$ is the "fake curvature" of a 2-bundle with 2-connection (or of a gerbe with connection and curving) which we will re-encounter in part III.

8.4 Appendix: Boundary state formalism

As a background for §8.1.2 (p.165) this section summarizes basic aspects of boundary conformal field theory (as discussed for instance in [162, 163, 164]).

Given a conformal field theory on the complex plane (with coordinates $z, \bar{z}$) we get an associated ('descendant') *boundary conformal field theory* (BCFT) on the upper half plane (UHP), $\mathrm{Im}(z) > 0$, by demanding suitable boundary condition on the real line. The only class of cases well understood so far is that where the chiral fields $W(z), \bar{W}(\bar{z})$ can be analytically continued to the real line $\mathrm{Im}(z) = 0$ and a local automorphism of the chiral algebra exists, the *gluing map* Ω, such that on the boundary the left- and right-moving fields are related by

$$W(z) = \Omega \bar{W}(\bar{z}) \,, \qquad \text{at } z = \bar{z} \,. \tag{8.86}$$

In particular Ω always acts trivially on the energy momentum current

$$\Omega \bar{T}(\bar{z}) = \bar{T}(\bar{z}) \tag{8.87}$$

so that

$$T(z) = \bar{T}(\bar{z}) \,, \qquad \text{at } z = \bar{z} \,, \tag{8.88}$$

which ensures that no energy-momentum flows off the boundary.

This condition allows to introduce for every chiral $W, \bar{W}$ the single chiral field

$$\mathrm{W}(z) = \begin{cases} W(z) & \text{for } \mathrm{Im}(z) \geq 0 \\ \Omega\big(\bar{W}\big)(\bar{z}) & \text{for } \mathrm{Im}(z) < 0 \end{cases} \tag{8.89}$$

defined in the entire plane. (This is known as the 'doubling trick'.)

Since it is relatively awkward to work with explicit constraints it is desirable to find a framework where the boundary condition on fields at the real line can be replaced by an operator insertion in a bulk theory without boundary.

Imagine an open string propagating with both ends attached to some D-brane. The worldsheet is topologically the disk (with appropriate operator insertions at the boundary). This disk can equivalently be regarded as the half sphere glued to the brane. But from this point of view it represents the worldsheet of a closed string with a certain source at the brane. Therefore the open string disk correlator on the brane is physically the same as a closed string emission from the brane with a certain source term corresponding to the open string boundary condition. The source term at the boundary of the half sphere can be represented by an operator insertion in the full sphere. The state corresponding to this vertex insertion is the *boundary state*.

In formal terms this heuristic picture translates to the following procedure:

First map the open string worldsheet to the sphere, in the above sense. By stereographic projection, the sphere is mapped to the plane and the upper half sphere which represents the open string worldsheet disk gets mapped to the complement of the unit disk in the plane. Denote the complex coordinates on this complement by $\zeta, \bar{\zeta}$ and let the open string worldsheet time $\tau = -\infty$ be mappped to $\zeta = 1$ and $\tau = +\infty$ mapped to $\zeta = -1$ (so that the open string propagates 'from right to left' in these worldsheet coordinates). With $z, \bar{z}$ the coordinates on the UHP this corresponds to $z = 0 \mapsto \zeta = 1$ and $z = \infty \mapsto \zeta = -1$. The rest of the boundary of the string must get mapped to the unit circle, which is where the string is glued to the brane. An invertible holomorphic map from the UHP to the complement of the unit disk with these features[11] is

$$\zeta(z) \equiv \frac{1 - iz}{1 + iz} \,. \tag{8.91}$$

For a given boundary condition α the boundary state $|\alpha\rangle$ is now defined as the state corresponding to the operator which, when inserted in the sphere, makes the correlator of some open string field Φ on the sphere equal to that on the UHP with boundary condition α:

$$\langle \Phi^{(\mathrm{H})}(z, \bar{z}) \rangle_\alpha = \left(\frac{\partial \zeta}{\partial z}\right)^h \left(\frac{\partial \bar{\zeta}}{\partial \bar{z}}\right)^{\bar{h}} \langle 0 | \, \Phi^{(\mathrm{P})}\left(\zeta, \bar{\zeta}\right) |\alpha\rangle \,. \tag{8.92}$$

Noting that on the boundary we have

$$\frac{\partial \zeta}{\partial z} = -i\zeta \,, \qquad \text{at } z = \bar{z} \Leftrightarrow \zeta = 1/\bar{\zeta} \tag{8.93}$$

the gluing condition (8.86) becomes in the new coordinates

$$\begin{aligned} \left(\frac{\partial \zeta}{\partial z}\right)^h W(\zeta) &= \left(\frac{\partial \bar{\zeta}}{\partial \bar{z}}\right)^h \Omega \bar{W}\left(\bar{\zeta}\right) \\ \Leftrightarrow W(\zeta) &= (-1)^h \bar{\zeta}^{2h} \, \Omega \bar{W}\left(\bar{\zeta}\right) \,, \qquad \text{at } \zeta = 1/\bar{\zeta} \,. \end{aligned} \tag{8.94}$$

In the theory living on the plane this condition translates into a constraint on the boundary state $|\alpha\rangle$:

$$\begin{aligned} 0 &\stackrel{!}{=} \langle 0 | \cdots \sum_{n=-\infty}^{\infty} \left(W_n \zeta^{-n-h} - (-1)^h \zeta^{-2h} \Omega \bar{W}_n \zeta^{n+h}\right) |\alpha\rangle \\ &= \langle 0 | \cdots \sum_{n=-\infty}^{\infty} \left(W_n \zeta^{-n-h} - (-1)^h \Omega \bar{W}_n \zeta^{n-h}\right) |\alpha\rangle \,, \qquad \forall \, \zeta = 1/\bar{\zeta} \,, \end{aligned} \tag{8.95}$$

i.e.

$$\left(W_n - (-1)^h \Omega \bar{W}_{-n}\right) |\alpha\rangle = 0 \,, \qquad \forall \, n \in \mathbb{N} \,. \tag{8.96}$$

---

[11]

$$|\zeta|^2 = \frac{1 + |z|^2 + 2\mathrm{Im}(z)}{1 + |z|^2 - 2\mathrm{Im}(z)} \geq 1 \quad \text{for } \mathrm{Im}(z) \geq 0 \tag{8.90}$$

Since $\Omega\bar{T} = \bar{T}$ holds for all BCFTs this implies in particular that one always has

$$\big(L_n - \bar{L}_{-n}\big)\,|\alpha\rangle = 0 \qquad \forall\, n\,, \tag{8.97}$$

which says that $|\alpha\rangle$ is invariant with respect to reparametrizations of the spatial worldsheet variable σ parameterizing the boundary (*cf.* for instance section 3 of [27]).

"A picture says
more than a thousand
words."

Part III

Nonabelian Strings

This part is concerned with categorified gauge theory. The concept of a categorified principal bundle with connection and holonomy is defined and investigated both in the "integral picture" and in the "differential picure" (*cf.* figure 7, p. 21). The key notion is that of global 2-holonomy (surface holonomy) which allows to construct "nonabelian strings."

Some basic concepts necessary for the following discussions are introduced in §9 (p.182) (see also the general introduction to categories in §4.3 (p.72)). This material is taken from the paper [31], written in collaboration with John Baez.

Then Lie 2-groups, Lie 2-algebras and loop groups are discussed in §10 (p.200), with an emphasis on how the group String(n) (*cf.* §4.2 (p.68)) is expressible in the language of 2-groups. This is taken from [32], which is a collaboration with John Baez, Alissa Crans and Danny Stevenson.

The analysis of 2-bundles and of 2-connections with 2-holonomy is begun in §11 (p.242), which is again taken from the work [31] with John Baez.

Buidling on that, globally defined 2-holonomy (surface holonomy) is defined and investigated in §12 (p.285). This is material from a paper in preparation [34].

The differential version of these considerations, which gives rise to generalized Deligne hypercohomology, is then discussed in §13 (p.335).

9. Preliminaries

To develop the theory of 2-connections, we need some mathematical preliminaries on internalization (§9.1 (p.182)), with special emphasis on Lie 2-groups and Lie 2-algebras (§9.3 (p.188)) and also 2-spaces (§9.2 (p.185)). We also review the theory of nonabelian gerbes (§9.4 (p.194)).

9.1 Internalization

To categorify concepts from differential geometry, we will use a procedure called 'internalization'. Developed by Lawvere, Ehresmann [165] and others, internalization is a method for generalizing concepts from ordinary set-based mathematics to other contexts — or more precisely, to other *categories*. This method is simple and elegant. To internalize a concept, we merely have to describe it using commutative diagrams in the category of sets, and then interpret these diagrams in some other category K. For example, if we internalize the concept of 'group' in the category of topological spaces, we obtain the concept of 'topological group'.

For categorification, the main concept we need to internalize is that of a category. To do this, we start by writing down the definition of category using commutative diagrams. We do this in terms of the functions s and t assigning to any morphism $f\colon x \to y$ its source and target:

$$s(f) = x, \qquad t(f) = y,$$

the function id assigning to any object its identity morphism:

$$\mathrm{id}(x) = 1_x,$$

and the function $\circ$ assigning to any composable pair of morphisms their composite:

$$\circ(f, g) = f \circ g$$

If we write $\mathrm{Ob}(C)$ for the set of objects and $\mathrm{Mor}(C)$ for the set of morphisms of a category C, the set of composable pairs of morphisms is denoted $\mathrm{Mor}(C)_s{\times}_t\mathrm{Mor}(C)$, since it consists of pairs (f, g) with $s(f) = t(g)$.

In these terms, the definition of category looks like this:

> A **small category**, say C, has a <u>set</u> of objects $\mathrm{Ob}(C)$, a <u>set</u> of morphisms $\mathrm{Mor}(C)$, source and target <u>functions</u>:
>
> $$s, t\colon \mathrm{Mor}(C) \to \mathrm{Ob}(C),$$
>
> an identity-assigning <u>function</u>:
>
> $$\mathrm{id}\colon \mathrm{Ob}(C) \to \mathrm{Mor}(C)$$
>
> and a composition <u>function</u>:
>
> $$\circ\colon \mathrm{Mor}(C)_s{\times}_t\mathrm{Mor}(C) \to \mathrm{Mor}(C)$$

> making diagrams commute that express associativity of composition, the left and right unit laws for identity morphisms, and the behaviour of source and target under composition.

We omit the actual diagrams because they are not very enlightening: the reader can find them elsewhere [40] or reinvent them. The main point here is not so much what they are, as that they *can* be written down.

To internalize this definition, we replace the word 'set' by 'object of K' and replace the word 'function' by 'morphism of K':

> A **category in** K, say C, has an object $\mathrm{Ob}(C) \in K$, an object $\mathrm{Mor}(C) \in K$, source and target morphisms:
>
> $$s, t\colon \mathrm{Ob}(C) \to \mathrm{Mor}(C),$$
>
> an identity-assigning morphism:
>
> $$\mathrm{id}\colon \mathrm{Ob}(C) \to \mathrm{Mor}(C),$$
>
> and a composition morphism:
>
> $$\circ\colon \mathrm{Mor}(C)_s{\times}_t\mathrm{Mor}(C) \to \mathrm{Mor}(C)$$
>
> making diagrams commute that express associativity of composition, the left and right unit laws for identity morphisms, and the behaviour of source and target under composition.

Here we must define $\mathrm{Mor}(C)_s{\times}_t\mathrm{Mor}(C)$ using a category-theoretic notion called a 'pullback' [89]. Luckily, in examples it is usually obvious what this pullback should be, since it consists of composable pairs of morphisms in C.

Using this method, we can instantly categorify various concepts used in gauge theory:

Definition 9.1. *A* **Lie 2-group** *is a category in* LieGrp*, the category whose objects are Lie groups and whose morphisms are smooth group homomorphisms.*

Definition 9.2. *A* **Lie 2-algebra** *is a category in* LieAlg*, the category whose objects are Lie algebras and whose morphisms are Lie algebra homomorphisms.*

(For the benefit of experts, we should admit that we are only defining 'strict' Lie 2-groups and Lie 2-algebras here.)

We could also define a 'smooth 2-space' to be a category in Diff, the category whose objects are finite-dimensional smooth manifolds and whose morphisms are smooth maps. However, this notion is slightly awkward for two reasons. First, unlike LieGrp and LieAlg, Diff does not have pullbacks in general. So, the subset of $\mathrm{Mor}(C) \times \mathrm{Mor}(C)$ consisting of composable pairs of morphisms may not be a submanifold. Second, and more importantly, we will also be interested in infinite-dimensional examples.

To solve these problems, we need a category of 'smooth spaces' that has pullbacks and includes a sufficiently large class of infinite-dimensional manifolds. Various categories of this sort have been proposed. It is unclear which one is best, but we shall use a slight variant of an idea proposed by Chen [166]. We call this category C^∞. For the present purposes, all that really matters about this category is that it has many nice features, including:

- Every finite-dimensional smooth manifold is a smooth space, with smooth maps between these being precisely those that are smooth in the usual sense.
- Every smooth space has a topology, and all smooth maps between smooth spaces are continuous.
- Every subset of a smooth space is a smooth space.
- Every quotient of a smooth space by an equivalence relation whose equivalence classes are closed subsets is a smooth space.
- If $\{X_\alpha\}_{\alpha\in A}$ are smooth spaces, so is their product $\prod_{\alpha\in A} X_\alpha$.
- If $\{X_\alpha\}_{\alpha\in A}$ are smooth spaces, so is their disjoint union $\coprod_{\alpha\in A} X_\alpha$.
- If X and Y are smooth spaces, so is the set $C^\infty(X,Y)$ consisting of smooth maps from X to Y.
- There is an isomorphism of smooth spaces
$$C^\infty(A\times X, Y) \cong C^\infty(A, C^\infty(X,Y))$$
sending any function $f\colon A\times X\to Y$ to the function $\hat{f}\colon A\to C^\infty(X,Y)$ given by $\hat{f}(x)(a) = f(x,a)$.
- We can define vector fields and differential forms on smooth spaces, with many of the usual properties.

With the notion of smooth space in hand, we can make the following definition:

Definition 9.3. *A* **(smooth) 2-space** *is a category in C^∞, the category whose objects are smooth spaces and whose morphisms are smooth maps.*

Not only can we categorify Lie groups, Lie algebras and smooth spaces, we can also categorify the maps between them. The right sort of map between categories is a functor: a pair of functions sending objects to objects and morphisms to morphisms, preserving source, target, identities and composition. If we internalize this concept, we get the definition of a 'functor in K'. We then say:

Definition 9.4. *A* **homomorphism** *between Lie 2-groups is a functor in* LieGrp.

Definition 9.5. *A* **homomorphism** *between Lie 2-algebras is a functor in* LieAlg.

Definition 9.6. *A* **(smooth) map** *between 2-spaces is a functor in* C^∞*.*

There are also natural transformations between functors, and internalizing this notion we can make the following definitions:

Definition 9.7. *A* **2-homomorphism** *between homomorphisms between Lie 2-groups is a natural transformation in* LieGrp*.*

Definition 9.8. *A* **2-homomorphism** *between homomorphisms between Lie 2-algebras is a natural transformation in* LieAlg*.*

Definition 9.9. *A* **(smooth) 2-map** *between maps between 2-spaces is a natural transformation in* C^∞*.*

Writing down these definitions is quick and easy. It takes longer to understand them and apply them to higher gauge theory. For this we must unpack them and look at examples. We do this in the next two sections.

9.2 2-Spaces

Unraveling (def. 9.3), a smooth 2-space, or **2-space** for short, is a category X where:

- The set of objects, $\mathrm{Ob}(X)$, is a smooth space.
- The set of morphisms, $\mathrm{Mor}(X)$, is a smooth space.
- The functions mapping any morphism to its source and target, $s, t\colon \mathrm{Mor}(X) \to \mathrm{Ob}(X)$, are smooth maps.
- The function mapping any object to its identity morphism, $\mathrm{id}\colon \mathrm{Ob}(X) \to \mathrm{Mor}(X)$, is a smooth map
- The function mapping any composable pair of morphisms to their composite, $\circ\colon \mathrm{Mor}(X)_s{\times}_t\mathrm{Mor}(X) \to \mathrm{Mor}(X)$, is a smooth map.

2-spaces are more common than one might at first guess. One only needs to know where to look. Here are some examples, working up to three that arise naturally in string theory: the path groupoid of M, the loop groupoid of M, and the 2-space of infinitesimal loops in M.

Definition 9.10. *A 2-space with only identity morphisms is called* **trivial***. A 2-space for which the source and target maps coincide is called* **simple***.*

Example 9.1. Any smooth space M gives a trivial 2-space X with $\mathrm{Ob}(X) = M$. This 2-space has $\mathrm{Mor}(X) = M$, with $s, t, i, \circ$ all being the identity map from M to itself. Every trivial 2-space is of this form.

Example 9.2. A **smooth monoid** is a smooth space with a smooth associative product and an identity element. Suppose that $E \xrightarrow{p} B$ is a smooth bundle of smooth monoids, not necessarily locally trivial. In other words, suppose that $E \xrightarrow{p} B$ is a map of smooth spaces, each fiber $p^{-1}(b)$ is equipped with an associative product and unit, and the fiberwise-defined product $\circ\colon E_p{\times}_pE \to E$ and the map $i\colon B \to E$ sending each point $b \in B$ to the identity element in its fiber are smooth. Then there is a simple 2-space X with $\mathrm{Ob}(X) = B$, $\mathrm{Mor}(X) = E$, $s = t = p$, and $i, \circ$ as above. Moreover, every simple 2-space is of this form.

Example 9.3. Given a smooth space M, there is a smooth 2-space $\mathcal{P}_1(M)$, the **path groupoid of** M, such that:

- the objects of $\mathcal{P}_1(M)$ are points of M,
- the morphisms of $\mathcal{P}_1(M)$ are thin homotopy classes of smooth paths $\gamma\colon [0,1] \to M$ such that $\gamma(t)$ is constant near $t = 0$ and $t = 1$.

Here a **thin homotopy** between smooth paths $\gamma_0, \gamma_1\colon [0,1] \to M$ is a smooth map $F\colon [0,1]^2 \to M$ such that:

- $F(0,t) = \gamma_0(t)$ and $F(1,t) = \gamma_1(t)$,
- $F(s,t)$ is constant for t near 0 and constant for t near 1,
- $F(s,t)$ is independent of s for s near 0 and for s near 1,
- the rank of the differential $dF(s,t)$ is ≤ 1 for all $(s,t) \in [0,1]^2$.

The last condition is what makes the homotopy 'thin': it guarantees that the homotopy sweeps out a surface of vanishing area.

To see how $\mathcal{P}_1(M)$ becomes a 2-space, first note that the space of smooth maps $\gamma\colon [0,1] \to M$ becomes a smooth space in a natural way, as does the subspace satisfying the constancy conditions near $t = 0, 1$, and finally the quotient of this subspace by the thin homotopy relation. This guarantees that $\mathrm{Mor}(\mathcal{P}_1(M))$ is a smooth space. For short, we call this smooth space PM, the **path space of** M. $\mathrm{Ob}(\mathcal{P}_1(M)) = M$ is obviously a smooth space as well. The source and target maps

$$s, t\colon \mathrm{Mor}(\mathcal{P}_1(M)) \to \mathrm{Ob}(\mathcal{P}_1(M))$$

send any equivalence class of paths to its endpoints:

$$s([\gamma]) = \gamma(0), \qquad t([\gamma]) = \gamma(1).$$

The identity-assigning map sends any point $x \in M$ to the constant path at this point. The composition map $\circ$ sends any composable pair of morphisms $[\gamma], [\gamma']$ to $[\gamma \circ \gamma']$, where

$$\gamma \circ \gamma'(t) = \begin{cases} \gamma(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \gamma'(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

One can check that $\gamma \circ \gamma'$ is a smooth path and that $[\gamma \circ \gamma']$ is well-defined and independent of the choice of representatives for $[\gamma]$ and $[\gamma']$. One can also check that the maps $s, t, i, \circ$ are smooth and that the usual rules of a category hold. It follows that $\mathcal{P}_1(M)$ is a 2-space.

In fact, $\mathcal{P}_1(M)$ is not just a category: it is also a **groupoid**: a category where every morphism has an inverse. The inverse of $[\gamma]$ is just $[\overline{\gamma}]$, where $\overline{\gamma}$ is obtained by reversing the orientation of the path γ:

$$\overline{\gamma}(t) = \gamma(1-t).$$

Moreover, the map sending any morphism to its inverse is smooth. Thus $\mathcal{P}_1(M)$ is a **smooth groupoid**: a 2-space where every morphism is invertible and the map sending every morphism to its inverse is smooth.

Example 9.4. Given a 2-space X, any subcategory of X becomes a 2-space in its own right. Here a **subcategory** is a category Y with $\mathrm{Ob}(Y) \subseteq \mathrm{Ob}(X)$ and $\mathrm{Mor}(Y) \subseteq \mathrm{Mor}(X)$, where the source, target, identity-assigning and composition maps of Y are restrictions of those for X. The reason Y becomes a 2-space is that any subspace of a smooth space becomes a smooth space in a natural way and restrictions of smooth maps to subspaces are smooth. We call Y a **sub-2-space** of X.

Example 9.5. Given a smooth space M, the path groupoid $\mathcal{P}_1(M)$ has a sub-2-space $\mathcal{L}M$ whose objects are all the points of M and whose morphisms are those equivalence classes $[\gamma]$ where γ is a loop: that is, a path with $\gamma(0) = \gamma(1)$. We call $\mathcal{L}M$ the **loop groupoid** of M. Like the path groupoid, the loop groupoid of M is not just a 2-space, but a smooth groupoid.

Example 9.6. Given a smooth vector bundle $E \xrightarrow{p} B$ over a smooth space B, there is a simple 2-space $\mathcal{E}$ with B as its space of objects, E as its space of morphisms, and addition within each fiber as the operation of composing morphisms. This is a special case of Example 9.2. Since every vector in each fiber of E has an additive inverse, $\mathcal{E}$ is actually a smooth groupoid.

In particular, if $E = \Lambda^2 TB$ is the bundle of antisymmetric rank $(2,0)$ tensors over the smooth space B, we call $\mathcal{E}$ the **2-space of infinitesimal loops in** B. We can think of this as a kind of 'limit' of the loop groupoid of M in which the loops shrink to zero size.

For 2-spaces, and indeed for all categorified concepts, the usual notion of 'isomorphism' is less useful than that of 'equivalence'. For example, in categorified gauge theory what matters is not 2-bundles whose fibers are all isomorphic to some standard fiber, but those whose fibers are all *equivalent* to some standard fiber. We recall the concept of equivalence here:

Definition 9.11. *Given 2-spaces X and Y, an* **isomorphism** *$f\colon X \to Y$ is a map equipped with a map $\bar{f}\colon Y \to X$ such that $\bar{f}f = 1_X$ and $f\bar{f} = 1_Y$. An* **equivalence** *$f\colon X \to Y$ is a map equipped with a map $\bar{f}\colon Y \to X$ and invertible 2-maps $\phi\colon \bar{f}f \Longrightarrow 1_X$ and $\bar{\phi}\colon f\bar{f} \Longrightarrow 1_Y$.*

9.3 Lie 2-Groups and Lie 2-Algebras

Unravelling (def. 9.1), we see that a Lie 2-group $\mathcal{G}$ is a category where:

- The set of objects, $\mathrm{Ob}(\mathcal{G})$, is a Lie group.
- The set of morphisms, $\mathrm{Mor}(\mathcal{G})$, is a Lie group.
- The functions mapping any morphism to its source and target, $s, t\colon \mathrm{Mor}(\mathcal{G}) \to \mathrm{Ob}(\mathcal{G})$, are homomorphisms.
- The function mapping any object to its identity morphism, $\mathrm{id}\colon \mathrm{Ob}(\mathcal{G}) \to \mathrm{Mor}(\mathcal{G})$, is a homomorphism.
- The function mapping any composable pair of morphisms to their composite, $\circ\colon \mathrm{Mor}(\mathcal{G})_s{\times}_t\mathrm{Mor}(\mathcal{G}) \to \mathrm{Mor}(\mathcal{G})$, is a homomorphism.

For applications to higher gauge theory it is suggestive to draw objects of $\mathcal{G}$ as arrows:

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g"] & \bullet \end{tikzcd}$$

and morphisms $f\colon g \to g'$ as surfaces of this sort:

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow f"] & \bullet \end{tikzcd}$$

This lets us can draw multiplication in $\mathrm{Ob}(\mathcal{G})$ as composition of arrows, multiplication in $\mathrm{Mor}(\mathcal{G})$ as 'horizontal composition' of surfaces, and composition of morphisms $f\colon g \to g'$ and $f'\colon g' \to g''$ as 'vertical composition' of surfaces.

In this notation, the fact that composition is a homomorphism says that the 'exchange law'

$$(f_1 \circ f_1')(f_2 \circ f_2') = (f_1 f_2) \circ (f_1' f_2')$$

holds whenever we have a situation of this sort:

$$\begin{tikzcd}[column sep=large] \bullet \arrow[r, bend left=60, "g_1"] \arrow[r, "g_2" description] \arrow[r, bend right=60, "g_3"'] \arrow[r, phantom, shift left=3ex, "\Downarrow f_1"] \arrow[r, phantom, shift right=3ex, "\Downarrow f_1'"] & \bullet \arrow[r, bend left=60, "g_1'"] \arrow[r, "g_2'" description] \arrow[r, bend right=60, "g_3'"'] \arrow[r, phantom, shift left=3ex, "\Downarrow f_2"] \arrow[r, phantom, shift right=3ex, "\Downarrow f_2'"] & \bullet \end{tikzcd}$$

In other words, we can interpret this picture either as a horizontal composite of vertical composites or a vertical composite of horizontal composites, without any ambiguity.

A Lie 2-group with only identity morphisms is the same thing as a Lie group. To get more interesting examples it is handy to think of a Lie 2-group as special sort of 'crossed module'. To do this, start with a Lie 2-group $\mathcal{G}$ and form the pair of Lie groups

$$G = \mathrm{Ob}(\mathcal{G}), \qquad H = \ker s \subseteq \mathrm{Mor}(\mathcal{G}).$$

The target map restricts to a homomorphism

$$t\colon H \to G.$$

Besides the usual action of G on itself by conjugation, there is also an action of G on H,

$$\alpha\colon G \to \mathrm{Aut}(H),$$

given by

$$\begin{aligned} \alpha(g)(h) &= 1_g\, h\, 1_{g^{-1}} \\ &= \begin{tikzcd} \bullet \arrow[r, bend left, "g"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow 1_g"] & \bullet \arrow[r, bend left, "1"] \arrow[r, bend right, "t(h)"'] \arrow[r, phantom, "\Downarrow h"] & \bullet \arrow[r, bend left, "g^{-1}"] \arrow[r, bend right, "g^{-1}"'] \arrow[r, phantom, "\Downarrow 1_{g^{-1}}"] & \bullet \end{tikzcd} . \end{aligned}$$

The target map is equivariant with respect to this action:

$$t(\alpha(g)(h)) = g\, t(h)\, g^{-1}$$

and satisfies the so-called 'Peiffer identity':

$$\alpha(t(h))(h') = hh'h^{-1}.$$

A setup like this with groups rather than Lie groups is called a 'crossed module', so here we are getting a 'Lie crossed module':

Definition 9.12. *A* **Lie crossed module** *is a quadruple* (G, H, t, α) *consisting of Lie groups* G *and* H*, a homomorphism* $t\colon H \to G$*, and an action of* G *on* H *(that is, a homomorphism* $\alpha\colon G \to \mathrm{Aut}(H)$*) such that* t *is equivariant:*

$$t(\alpha(g)(h)) = g\, t(h)\, g^{-1}$$

and satisfies the Peiffer identity:

$$\alpha(t(h))(h') = hh'h^{-1}$$

for all $g \in G$ *and* $h, h' \in H$*.*

This definition becomes a bit more memorable if we abuse language and write $\alpha(g)(h)$ as ghg^{-1}; then the equations above become

$$t(ghg^{-1}) = g\, t(h)\, g^{-1}$$

and

$$t(h)\, h'\, t(h)^{-1} = hh'h^{-1}.$$

As we shall see, Lie 2-groups are essentially the same as Lie crossed modules. The same is true for the homomorphisms between them. We have already defined a homomorphism of Lie 2-groups as a functor in LieGrp. We can also define a homomorphism of Lie crossed modules:

Definition 9.13. *A* **homomorphism** *from the Lie crossed module (G, H, t, α) to the Lie crossed module (G', H', t', α') is a pair of homomorphisms $f\colon G \to G'$, $\tilde{f}\colon H \to H'$ such that*

$$t(\tilde{f}(h)) = f(t'(h))$$

and

$$\tilde{f}(\alpha(g)(h)) = \alpha'(f(g))(\tilde{f}(h))$$

for all $g \in G$, $h \in H$.

Not only does every Lie 2-group give a Lie crossed module; every Lie crossed module gives a Lie 2-group. In fact:

Proposition 9.1. *The category of Lie 2-groups is equivalent to the category of Lie crossed modules.*

Proof. This follows easily from the well-known equivalence between crossed modules and 2-groups [167]; details can also be found in [41]. For the convenience of the reader, we sketch how to recover a Lie 2-group from a Lie crossed module.

Suppose we have a Lie crossed module (G, H, t, α). Let

$$\mathrm{Ob}(\mathcal{G}) = G, \qquad \mathrm{Mor}(\mathcal{G}) = G \ltimes H$$

where the semidirect product is formed using the action of G on H, so that multiplication in $\mathrm{Mor}(\mathcal{G})$ is given by:

$$(g, h)(g', h') = (gg', h\alpha(g)(h')). \tag{9.1}$$

The inverse of an element of $\mathrm{Mor}(\mathcal{G})$ is given by:

$$(g, h)^{-1} = \left(g^{-1}, \alpha\left(g^{-1}\right)\left(h^{-1}\right)\right).$$

We make this into a Lie 2-group where the source and target maps $s, t\colon \mathrm{Mor}(\mathcal{G}) \to \mathrm{Ob}(\mathcal{G})$ are given by

$$s(g, h) = g, \qquad t(g, h) = t(h)g, \tag{9.2}$$

the identity-assigning map $\mathrm{id}\colon \mathrm{Ob}(\mathcal{G}) \to \mathrm{Mor}(\mathcal{G})$ is given by

$$\mathrm{id}(g) = (g, 1), \tag{9.3}$$

and the composite of the morphisms

$$(g, h)\colon g \to g', \qquad (g', h')\colon g' \to g'',$$

is

$$(g, h) \circ (g', h') = (g, h'h)\colon g \to g''. \tag{9.4}$$

It is also worth noting that every morphism has an inverse with respect to composition, which we denote by

$$\overline{(g, h)} = \left(t(h)\, g, h^{-1}\right).$$

One can check that this construction indeed gives a Lie 2-group, and that together with the previous construction it sets up an equivalence between the categories of Lie 2-groups and Lie crossed modules. $\square$

Crossed modules are important in homotopy theory [168], and the reader who is fonder of crossed modules than categories is free to think of Lie 2-groups as a way of talking about Lie crossed modules. Both perspectives are useful, but one advantage of Lie crossed modules is that they allow us to quickly describe some examples:

Example 9.7. Given any abelian group H, there is a Lie crossed module where G is the trivial group and t, α are trivial. This gives a Lie 2-group $\mathcal{G}$ with one object and H as the group of morphisms. Lie 2-groups of this sort are important in the theory of *abelian* gerbes.

Example 9.8. More generally, given any Lie group G, abelian Lie group H, and action α of G as automorphisms of H, there is a Lie crossed module with $t\colon G \to H$ the trivial homomorphism. For example, we can take H to be a finite-dimensional vector space and α to be a representation of G on this vector space.

In particular, if G is the Lorentz group and α is the defining representation of this group on Minkowski spacetime, this construction gives a Lie 2-group called the **Poincaré 2-group**, because its group of morphisms is the Poincaré group. After its introduction in work on higher gauge theory [21], this 2-group has come to play an important role in some recent work on quantum gravity by Crane, Sheppeard and Yetter [169, 170].

Example 9.9. Given any Lie group H, there is a Lie crossed module with $G = \mathrm{Aut}(H)$, $t\colon H \to G$ the homomorphism assigning to each element of H the corresponding inner automorphism, and the obvious action of G as automorphisms of H. We call the corresponding Lie 2-group the **automorphism 2-group** of H, and denote it by $\mathcal{AUT}(H)$. This sort of 2-group is important in the theory of *nonabelian* gerbes.

In particular, if we take H to be the multiplicative group of nonzero quaternions, then $G = \mathrm{SU}(2)$ and we obtain a 2-group that plays a basic role in Thompson's theory of quaternionic gerbes [171].

We use the term 'automorphism 2-group' because $\mathcal{AUT}(H)$ really is a 2-group of symmetries of H. An object of $\mathcal{AUT}(H)$ is a symmetry of the group H in the usual sense: that is, an automorphism $f\colon H \to H$. On the other hand, a morphism $\theta\colon f \to f'$ in $\mathcal{AUT}(H)$ is a 'symmetry between symmetries': that is, an element $h \in H$ that sends f to f' in the following sense: $hf(x)h^{-1} = f'(x)$ for all $x \in H$.

Example 9.10. Suppose that $1 \to A \hookrightarrow H \xrightarrow{t} G \to 1$ is a central extension of the Lie group G by the Lie group H. Then there is a Lie crossed module with this choice of $t\colon G \to H$. To construct α we pick any section s, that is, any function $s\colon G \to H$ with $t(s(g)) = g$, and define

$$\alpha(g)h = s(g)hs(g)^{-1}.$$

Since A lies in the center of H, α independent of the choice of s. We do not need a global smooth section s to show $\alpha(g)$ depends smoothly on g; it suffices that there exist a local smooth section in a neighborhood of each $g \in G$.

It is easy to generalize this idea to infinite-dimensional examples, like central extensions of loop groups, if we work not with Lie groups but **smooth groups**: that is, groups in the category of smooth spaces. The basic theory of smooth 2-groups and smooth crossed modules works just like the finite-dimensional case, but with the category of smooth spaces replacing Diff.

In particular, given a simply-connected connected simple Lie group G, the loop group ΩG is a smooth group. For each level $k \in \mathbb{Z}$, this group has a central extension

$$1 \to \mathrm{U}(1) \hookrightarrow \widehat{\Omega_k G} \xrightarrow{t} \Omega G \to 1$$

as explained by Pressley and Segal [172]. The above diagram lives in the category of smooth groups, and there exist local smooth sections for $t\colon \tilde{L}G \to LG$, so we obtain a smooth crossed module $(\Omega G, \widehat{\Omega_k G}, t, \alpha)$ with α given as above. This in turn gives a smooth 2-group which we call the **loop 2-group** of G, $\mathcal{L}_k G$. It has recently been shown [32] (see §10 (p.200)) that this fits into an exact sequence of smooth 2-groups:

$$1 \to \mathcal{L}_k G \hookrightarrow \mathcal{P}_k G \xrightarrow{t} G \to 1$$

where the middle term, the **path 2-group** of G, has extremely interesting properties. In particular, it gives a new construction of the group $\mathrm{String}(n)$ when $G = \mathrm{Spin}(n)$. So, we expect that these 2-groups $\mathcal{P}_k G$ will be especially interesting for applications of 2-bundles to string theory.

Just as Lie groups give rise to Lie algebras, Lie 2-groups give rise to Lie 2-algebras. These can also be described using a differential version of crossed modules. Recall that a Lie 2-algebra is a category $\mathcal{L}$ where:

- The set of objects, $\mathrm{Ob}(\mathcal{L})$, is a Lie algebra.
- The set of morphisms, $\mathrm{Mor}(\mathcal{L})$, is a Lie algebra.
- The functions mapping any morphism to its source and target, $s, t\colon \mathrm{Mor}(\mathcal{L}) \to \mathrm{Ob}(\mathcal{L})$, are Lie algebra homomorphisms,
- The function mapping any object to its identity morphism, $\mathrm{id}\colon \mathrm{Ob}(\mathcal{L}) \to \mathrm{Mor}(\mathcal{L})$, is a Lie algebra homomorphism.
- The function mapping any composable pair of morphisms to their composite, $\circ\colon \mathrm{Mor}(\mathcal{L})_s{\times}_t\mathrm{Mor}(\mathcal{L}) \to \mathrm{Mor}(\mathcal{L})$, is a Lie algebra homomorphism.

We can get a Lie 2-algebra by differentiating all the data in a Lie 2-group. Similarly, we can get a 'differential crossed module' by differentiating all the data in a Lie crossed module:

Definition 9.14. *A* **differential crossed module** *is a quadruple* $\mathcal{C} = (\mathfrak{g}, \mathfrak{h}, dt, d\alpha)$ *consisting of Lie algebras* $\mathfrak{g}, \mathfrak{h}$, *a homomorphism* $dt\colon \mathfrak{h} \to \mathfrak{g}$, *and an action* α *of* $\mathfrak{g}$ *as derivations of* $\mathfrak{h}$ *(that is, a homomorphism* $\alpha\colon \mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$*) satisfying*

$$dt(d\alpha(x)(y)) = [x, dt(y)] \tag{9.5}$$

and

$$d\alpha(dt(y))(y') = [y, y'] \tag{9.6}$$

for all $x \in \mathfrak{g}$ *and* $y, y' \in \mathfrak{h}$.

This definition becomes easier to remember if we allow ourselves to write $d\alpha(x)(y)$ as $[x, y]$. Then the fact that $d\alpha$ is an action of $\mathfrak{g}$ as derivations of $\mathfrak{h}$ simply means that $[x, y]$ is linear in each argument and the following 'Jacobi identities' hold:

$$[[x, x'], y] = [x, [x', y]] - [x', [x, y]], \tag{9.7}$$

$$[x, [y, y']] = [[x, y], y'] - [[x, y'], y] \tag{9.8}$$

for all $x, x' \in \mathfrak{g}$ and $y, y' \in \mathfrak{h}$. Furthermore, the two equations in the above definition become

$$t([x, y]) = [x, t(y)] \tag{9.9}$$

and

$$[t(y), y'] = [y, y']. \tag{9.10}$$

Proposition 9.2. *The category of Lie 2-algebras is equivalent to the category of differential crossed modules.*

Proof. The proof is just like that of Prop. 9.1. □

Since every Lie 2-group gives a Lie 2-algebra and a differential crossed module, we get plenty of examples of the latter concepts from our example of Lie 2-groups. Here is another interesting class of examples:

Example 9.11. Just as every Lie 2-group gives rise to a Lie 2-algebra, so does every smooth 2-group. The reason is that not only smooth manifolds but also smooth spaces have tangent spaces, and the usual construction of Lie algebras from Lie groups generalizes to smooth groups. So, any smooth 2-group $\mathcal{G}$ gives a Lie 2-algebra $\mathcal{L}$ for which $\mathrm{Ob}(\mathcal{L})$ is the Lie algebra of $\mathrm{Ob}(\mathcal{G})$, $\mathrm{Mor}(\mathcal{L})$ is the Lie algebra of $\mathrm{Mor}(\mathcal{G})$, and the maps $s, t, i, \circ$ for $\mathcal{L}$ are obtained by differentiating the corresponding maps for $\mathcal{G}$.

In particular, suppose G is a simply-connected compact simple Lie group with Lie algebra $\mathfrak{g}$. Then the loop 2-group of G, as defined in Example 9.10, has a Lie 2-algebra. This Lie 2-algebra has $L\mathfrak{g} = C^\infty(S^1, \mathfrak{g})$ as its Lie algebra of objects, and a certain central extension $\tilde{L}\mathfrak{g}$ of $L\mathfrak{g}$ as its Lie algebra of morphisms. We call this Lie 2-algebra the **loop**

Lie 2-algebra of $\mathfrak{g}$. It is another way of organizing the data in the affine Lie algebra corresponding to $\mathfrak{g}$.

Alternatively, we can take this central extension of smooth groups:

$$1 \to \mathrm{U}(1) \hookrightarrow \tilde{L}G \xrightarrow{t} LG \to 1$$

and differentiate all the maps to obtain a central extension of Lie algebras:

$$1 \to \mathfrak{u}(1) \hookrightarrow \tilde{L}\mathfrak{g} \xrightarrow{dt} L\mathfrak{g} \to 1.$$

Just as every central extension of Lie groups gives a Lie 2-group, every central extension of Lie algebras gives a Lie 2-algebra. So, we obtain a Lie 2-algebra, which is the loop Lie 2-algebra of $\mathfrak{g}$.

We conclude the preliminaries with a brief review of nonabelian gerbes.

9.4 Nonabelian Gerbes

Given a bundle $E \xrightarrow{p} M$, the sections of E defined on all possible open sets of B are naturally organized into a structure called a 'sheaf'. This codifies the fact that we can restrict a section from an open set $U \subseteq B$ to a smaller open set $U' \subseteq U$, and also piece together sections on open sets U_i covering U to obtain a unique section on U, as long as the sections agree on the intersections $U_i \cap U_j$. While mathematical physicists tend to be more familiar with bundles than sheaves, the greater generality of sheaves is important in algebraic geometry.

While the 2-bundles to be discussed in the present paper arise from categorifying the concept of 'bundle', most previous work on this subject starts by categorifying the concept of 'sheaf' to obtain the concept of 'stack', with 'gerbes' as a key special case. We suspect that just as mathematical physicists are more comfortable with bundles than sheaves, they will eventually prefer 2-bundles to gerbes. At present, however, it is crucial to clarify the relation between 2-bundles and gerbes. So, one of the goals of this paper is to relate 2-connections on 2-bundles to the already established notion of connections on gerbes. We begin here by recalling the history of stacks and gerbes, and the concept of a gerbe with connection.

The idea of a stack goes back to Grothendieck [173]. Just as a sheaf over a space M assigns a *set* of sections to any open set $U \subseteq M$, a stack assigns a *category* of sections to any open set $U \subseteq M$. Indeed, one may crudely define a stack as a 'sheaf of categories'. However, all the usual sheaf axioms need to be 'weakened', meaning that instead of equations between objects, we must use isomorphisms satisfying suitable equations of their own. For example, in a sheaf we can obtain a section s over U from sections s_i over open sets U_i covering U when these sections are *equal* on double intersections:

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$$

For a stack, on the other hand, we can obtain a section s over U when the sections s_i are *isomorphic* over double intersections:

$$h_{ij} \colon s_i|_{U_i \cap U_j} \xrightarrow{\sim} s_j|_{U_i \cap U_j},$$

as long as the isomorphisms satisfy the familiar 'cocycle condition' on triple intersections: $h_{ij}h_{jk} = h_{ik}$ on $U_i \cap U_j \cap U_k$.

A good example is the stack of principal H-bundles over M, where H is any fixed Lie group. This associates to each open set $U \subseteq M$ the category whose objects are principal H-bundles over U and whose morphisms are H-bundle isomorphisms. The above cocycle condition is very familiar in this case: it says when we can build a H-bundle s over U by gluing together H-bundles s_i over open sets covering U, using H-valued transition functions h_{ij} defined on double intersections.

This example also motivates the notion of a 'gerbe', which is a special sort of stack introduced by Giraud [174, 175]. For a stack over M to be a gerbe, it must satisfy three properties:

- Its category of sections over any open set must be a *groupoid*: that is, a category where all the morphisms are invertible.
- Each point of M must have a neighborhood over which the groupoid of sections is nonempty.
- Given two sections s, s' over an open set $U \subseteq M$, each point of U must have a neighborhood $V \subseteq U$ such that $s|_V \cong s'|_V$.

It is easy to see that the stack of principal H-bundles satisfies all these conditions. It satisfies another condition as well: for any section s over an open set $U \subseteq M$, each point of U has a neighborhood V such that the automorphisms of $s|_V$ form a group isomorphic to the group of smooth H-valued functions on V. A gerbe of this sort is called an 'H-gerbe'. Sometimes these are called 'nonabelian gerbes', to distinguish them from another class of gerbes that only make sense when the group H is abelian.

There is a precise sense in which the gerbe of principal H-bundles is the 'trivial' H-gerbe. Every H-gerbe is *locally* equivalent to this one, but not globally. So, we can think of a H-gerbe as a thing whose sections look locally like principal H-bundles, but not globally. This viewpoint is emphasized by the concept of 'bundle gerbe', defined first in the abelian case by Murray [176, 177] and more recently in the nonabelian case that concerns us here by Aschieri, Cantini and Jurčo [50].

However, the most concrete way of getting our hands on H-gerbes over M is by gluing together trivial H-gerbes defined on open sets U_i that cover M. This leads to a simple description of H-gerbes in terms of transition functions satisfying cocycle conditions. Now the transition functions defined on double intersections take values not in H but in $G = \mathrm{Aut}(H)$:

$$g_{ij} \colon U_i \cap U_j \to G$$

Moreover, they need not satisfy the usual cocycle condition for triple intersections 'on the nose', but only up to conjugation by certain functions

$$h_{ijk} \colon U_i \cap U_j \cap U_k \to H.$$

In other words, we demand:

$$g_{ij}g_{jk} = t(h_{ijk})\, g_{ik}$$

where $t\colon H \to G$ sends $h \in H$ to the operation of conjugating by h. Finally, the functions h_{ijk} should satisfy an cocycle condition on quadruple intersections:

$$\alpha(g_{ij})(h_{jkl})\ h_{ijl} = h_{ijk}h_{ikl}$$

where α is the natural action of $G = \mathrm{Aut}(H)$ on H. All this can be formalized most clearly using the automorphism 2-group $\mathcal{AUT}(H)$ described in Example 9.9, since this Lie 2-group has (G, H, t, α) as its corresponding Lie crossed module. Indeed, one way that 2-bundles generalize gerbes is by letting an arbitrary Lie 2-group play the role that $\mathcal{AUT}(H)$ plays here; we call this 2-group the 'structure 2-group' of the 2-bundle.

Given an H-gerbe, we can specify a 'connection' on it by means of some additional local data. We begin by choosing $\mathfrak{g}$-valued 1-forms A_i on the open sets U_i, which describe parallel transport along paths. But these 1-forms need not satisfy the usual consistency condition on double intersections! Instead, they satisfy it only up to $\mathfrak{h}$-valued 1-forms a_{ij}:

$$A_i + dt(a_{ij}) = g_{ij}A_j g_{ij}^{-1} + g_{ij}\mathbf{d}g_{ij}^{-1}.$$

These, in turn, must satisfy a consistency condition on triple intersections:

$$a_{ij} + \alpha(g_{ij})(a_{jk}) = h_{ijk}a_{ik}h_{ijk}^{-1} + h_{ijk}\, d\alpha(A_i)(h_{ijk}^{-1}).$$

Next, we choose $\mathfrak{h}$-valued 2-forms B_i describing parallel transport along surfaces. These satisfy a consistency condition on double intersections:

$$\alpha(g_{ij})(B_j) = B_i - k_{ij} + b_{ij}\ ,$$

where the $\mathfrak{h}$-valued 2-forms b_{ij} and

$$k_{ij} \equiv \mathbf{d}a_{ij} + a_{ij} \wedge a_{ij} - d\alpha(A_i) \wedge a_{ij}$$

measure the failure of B_i to transform covariantly. The 2-form k_{ij} is essentially the curvature of a_{ij}, while b_{ij} is a new object which turns out to have a transition law of its own, this time on triple intersections:

$$b_{ij} + \alpha(g_{ij})(d_{jk}) = h_{ijk}\, b_{ik}\, h_{ijk}^{-1} + h_{ijk}\, d\alpha(dt(B_i) + F_{A_i})\,(h_{ijk}^{-1})\,.$$

This description of connections on nonabelian gerbes was first given by Breen and Messing [49]. Aschieri, Cantini and Jurčo then gave a similar treatment using bundle gerbes [50].

Later, Aschieri and Jurčo [23] introduced connections on so-called 'twisted' nonabelian gerbes. These are a categorified version of the somewhat more familiar 'twisted bundles', so let us first recall the latter. Suppose the group H has a $\mathrm{U}(1)$ subgroup in its center, giving a central extension

$$1 \to \mathrm{U}(1) \hookrightarrow H \to H/\mathrm{U}(1) \to 1.$$

Then we can build a 'twisted H-bundle' using H-valued transition functions h_{ij} that only satisfy the usual cocycle condition on triple intersections up to a phase: $h_{ij}h_{jk} =$

$\lambda_{ijk} h_{ik}$. These phases automatically satisfy a cocycle condition on quadruple intersections: $\lambda_{ijk}\lambda_{ikl} = \lambda_{jkl}\lambda_{ijl}$.

Incidentally, this is just the cocycle condition for the transition function in an abelian gerbe. Since a bundle can be regarded as a 0-gerbe, this is an example of a general pattern in which the twist of a nonabelian twisted p-gerbe defines an abelian $(p+1)$-gerbe. This is called the 'lifting $(p+1)$-gerbe' because precisely when it is trivial does the obstruction to lifting the structure group of the p-gerbe to its central extension vanish. In fact, we can regard a principal U(1) bundle itself as an abelian 0-gerbe. This can be thought of as measuring the 'twist' of a (-1)-gerbe, which is just an ordinary H-valued function when the twist vanishes.

Following these considerations we can define a 'twisted H-gerbe' by relaxing the aforementioned cocycle condition on the functions h_{ijk}, requiring only that it hold up to a phase:

$$\alpha(g_{ij})(h_{jkl})\; h_{ijl} = \lambda_{ijkl}\, h_{ijk}\, h_{ikl}$$

where

$$\lambda_{ijkl} \colon U_i \cap U_j \cap U_k \cap U_l \to \mathrm{U}(1).$$

These phases automatically satisfy a cocycle condition on quintuple intersections,

$$\lambda_{ijkl}\lambda_{ijlm}\lambda_{jklm} = \lambda_{iklm}\lambda_{ijkm}\,,$$

and this is indeed precisely the cocycle condition for an abelian 2-gerbe.

In an analogous way, the cocycle conditions for the connection a_{ij} and curving B_i of a nonabelian gerbe can also pick up a twist. This amounts to adding Lie$(U(1))$-valued objects α_{ijk} and β_{ij} to the previous equations:

$$\begin{aligned}
\alpha_{ijk} &= a_{ij} + \alpha(g_{ij})(a_{jk}) - h_{ijk}\, a_{ik}\, h_{ijk}^{-1} - h_{ijk}\, d\alpha(A_i)(h_{ijk}^{-1}) \\
\beta_{ij} &= b_{ij} + \alpha(g_{ij})(d_{jk}) - h_{ijk}\, b_{ik}\, h_{ijk}^{-1} - h_{ijk}\, d\alpha(dt(B_i) - F_{A_i})\,(h_{ijk}^{-1})\,.
\end{aligned}$$

From these equations it follows that the twists λ_{ijkl}, α_{ijk} and β_{ij} themselves satisfy cocycle conditions that identify them as the transition functions, connection and curving of a U(1) 2-gerbe, the 'lifting 2-gerbe'.

The phenomenon of lifting gerbes has of course its analogue in the language of 2-bundles, but this will not concern us here.

To summarize, we list the local data for a twisted nonabelian gerbe with connection. For maximum generality, we start with an arbitrary Lie 2-group $\mathcal{G}$ and form its Lie crossed module (G, H, α, t). The definition below reduces to that of Aschieri and Jurčo when $\mathcal{G} = \mathcal{AUT}(H)$ and the 'phases' λ_{ijkl} lie in a chosen U(1) subgroup of the center of H. In general, we merely require these 'phases' to lie in the kernel of $t\colon H \to G$. This kernel always lies in the center of H, since if $h \in \ker t$ and $h' \in H$, the Peiffer identity gives

$$hh'h^{-1} = \alpha(t(h))(h') = h'.$$

So, all of Aschieri and Jucčo's calculations which require the phases to commute with other elements of H still go through.

Definition 9.15. *A* **twisted nonabelian gerbe with connection** *consists of:*

- *a base space* M*,*
- *an open cover* U *of* M*, with* $U^{[n]}$ *denoting the union of all* n*-fold intersections of patches in* U*,*
- *a Lie 2-group* $\mathcal{G}$ *with Lie crossed module* (G, H, α, t) *and differential crossed module* $(\mathfrak{g}, \mathfrak{h}, d\alpha, dt)$*,*
- *transition functions:*

$$\begin{aligned} g\colon U^{[2]} &\to G \\ (x, i, j) &\mapsto g_{ij}(x) \in G \end{aligned} \tag{9.11}$$

- *transition transformation functions:*

$$\begin{aligned} h\colon U^{[3]} &\to H \\ (x, i, j, k) &\mapsto h_{ijk}(x) \in H \end{aligned} \tag{9.12}$$

- *connection 1-forms:*

$$\begin{aligned} A &\in \Omega^1(U^{[1]}, \mathfrak{g}) \\ (x, i) &\mapsto A_i(x) \in \mathfrak{g} \end{aligned} \tag{9.13}$$

- *curving 2-forms:*

$$\begin{aligned} B &\in \Omega^2(U^{[1]}, \mathfrak{h}) \\ (x, i) &\mapsto B_i(x) \in \mathfrak{h} \end{aligned} \tag{9.14}$$

- *connection transformation 1-forms:*

$$\begin{aligned} a &\in \Omega^1(U^{[1]}, \mathfrak{h}) \\ (x, i, j) &\mapsto a_{ij}(x) \end{aligned} \tag{9.15}$$

- *curving transformation 2-forms:*

$$\begin{aligned} d &\in \Omega^2(U^{[2]}, \mathfrak{h}) \\ (x, i, j) &\mapsto d_{ij}(x) \end{aligned} \tag{9.16}$$

- *phases twisting the cocycle condition for the* h_{ijk}*:*

$$\begin{aligned} \lambda\colon U^{[4]} &\to \ker(t) \subseteq H \\ (x, i, j, k, l) &\mapsto \lambda_{ijkl}(x) \end{aligned} \tag{9.17}$$

- *phases twisting the cocycle condition for the* a_{ij}

$$\begin{aligned} \alpha \;&\in\; \Omega^1(U^{[3]}, \ker(dt)) \\ (x,i,j,k) \;&\mapsto\; \alpha_{ijk}(x) \end{aligned} \tag{9.18}$$

- *phases twisting the cocycle condition for the* b_{ij}

$$\begin{aligned} \beta \;&\in\; \Omega^2(U^{[2]}, \ker(dt)) \\ (x,i,j) \;&\mapsto\; \beta_{ij}(x) \end{aligned} \tag{9.19}$$

-

$$\begin{aligned} \gamma \;&\in\; \Omega^3(U^{[1]}, \ker(dt)) \\ (x,i) \;&\mapsto\; \gamma_i(x) \end{aligned} \tag{9.20}$$

such that the following cocycle conditions are satisfied:

- *cocycle condition for the* g_{ij}*:*

$$g_{ij}g_{jk} \;=\; t(h_{ijk})\, g_{ik} \tag{9.21}$$

- *cocycle condition for the* A_i*:*

$$A_i + dt(a_{ij}) \;=\; g_{ij}A_j g_{ij}^{-1} + g_{ij}\mathbf{d}g_{ij}^{-1} \tag{9.22}$$

- *cocycle condition for the* B_i*:*

$$B_i \;=\; \alpha(g_{ij})(B_j) + k_{ij} - d_{ij} - \beta_{ij}\,. \tag{9.23}$$

  where

$$k_{ij} \equiv \mathbf{d}a_{ij} + a_{ij}\wedge a_{ij} - d\alpha(A_i)\wedge a_{ij} \tag{9.24}$$

- *cocycle condition for the* d_{ij}*:*

$$d_{ij} + g_{ij}(d_{jk}) \;=\; h_{ijk}\, d_{ik}\, h_{ijk}^{-1} + h_{ijk}d\alpha(dt(B_i) + F_{A_i})\, h_{ijk}^{-1} \tag{9.25}$$

- *cocycle condition for the* h_{ijk}*:*

$$\alpha(g_{ij})(h_{jkl})\; h_{ijl} \;=\; \lambda_{ijkl}\, h_{ijk}\, h_{ikl} \tag{9.26}$$

- *cocycle condition for the* a_{ij}*:*

$$\alpha_{ijk} \;=\; a_{ij} + g_{ij}(a_{jk}) - h_{ijk}\, a_{ik}\, h_{ijk}^{-1} - h_{ijk}\,\mathbf{d}h_{ijk}^{-1} - h_{ijk}\, d\alpha(A_i)\,(h_{ijk}^{-1})\,. \tag{9.27}$$

Finally, the *curvature 3-form* of the nonabelian gerbe is defined as

$$H_i \equiv d_{A_i}B_i + \gamma_i, \tag{9.28}$$

and its transformation law is:

$$H_i \;=\; \phi_{ij}(H_j) - \mathbf{d}d_{ij} - [a_{ij}, d_{ij}] - d\alpha(dt(B_i) + F_{A_i})(a_{ij}) - d\alpha(A_i)(d_{ij})\,. \tag{9.29}$$

10. 2-Groups, Loop Groups and the String-group

The following is taken from [32], which is joint work with John Baez, Alissa Crans and Danny Stevenson.

10.1 Introduction

We describe an interesting relation between Lie 2-algebras, the Kac–Moody central extensions of loop groups, and the group $\mathrm{String}(n)$. A Lie 2-algebra is a categorified version of a Lie algebra where the Jacobi identity holds up to a natural isomorphism called the 'Jacobiator'. Similarly, a Lie 2-group is a categorified version of a Lie group. If G is a simply-connected compact simple Lie group, there is a 1-parameter family of Lie 2-algebras $\mathfrak{g}_k$ each having $\mathfrak{g}$ as its Lie algebra of objects, but with a Jacobiator built from the canonical 3-form on G. There appears to be no Lie 2-group having $\mathfrak{g}_k$ as its Lie 2-algebra, except when $k = 0$. Here, however, we construct for integral k an infinite-dimensional Lie 2-group $\mathcal{P}_k G$ whose Lie 2-algebra is *equivalent* to $\mathfrak{g}_k$. The objects of $\mathcal{P}_k G$ are based paths in G, while the automorphisms of any object form the level-k Kac–Moody central extension of the loop group ΩG. This 2-group is closely related to the kth power of the canonical gerbe over G. Its nerve gives a topological group $|\mathcal{P}_k G|$ that is an extension of G by $K(\mathbb{Z}, 2)$. When $k = \pm 1$, $|\mathcal{P}_k G|$ can also be obtained by killing the third homotopy group of G. Thus, when $G = \mathrm{Spin}(n)$, $|\mathcal{P}_k G|$ is none other than $\mathrm{String}(n)$.

The theory of simple Lie groups and Lie algebras has long played a central role in mathematics. Starting in the 1980s, a wave of research motivated by physics has revitalized this theory, expanding it to include structures such as quantum groups, affine Lie algebras, and central extensions of loop groups. All these structures rely for their existence on the left-invariant closed 3-form ν naturally possessed by any compact simple Lie group G:

$$\nu(x, y, z) = \langle x, [y, z] \rangle \qquad x, y, z \in \mathfrak{g},$$

or its close relative, the left-invariant closed 2-form ω on the loop group ΩG:

$$\omega(f, g) = 2 \int_{S^1} \langle f(\theta), g'(\theta) \rangle \, d\theta \qquad f, g \in \Omega\mathfrak{g}.$$

Moreover, all these new structures fit together in a grand framework that can best be understood with ideas from physics — in particular, the Wess–Zumino–Witten model and Chern–Simons theory. Since these ideas arose from work on string theory, which replaces point particles by higher-dimensional extended objects, it is not surprising that their study uses concepts from higher-dimensional algebra, such as gerbes [178, 179, 180].

More recently, work on higher-dimensional algebra has focused attention on Lie 2-groups [41] and Lie 2-algebras [40]. A '2-group' is a category equipped with operations analogous to those of a group, where all the usual group axioms hold only up to specified natural isomorphisms satisfying certain coherence laws of their own. A 'Lie 2-group' is a 2-group where the set of objects and the set of morphisms are smooth manifolds, and all the operations and natural isomorphisms are smooth. Similarly, a 'Lie 2-algebra' is a category equipped with operations analogous to those of a Lie algebra, satisfying the usual laws

up to coherent natural isomorphisms. Just as Lie groups and Lie algebras are important in gauge theory, Lie 2-groups and Lie 2-algebras are important in 'higher gauge theory', which describes the parallel transport of higher-dimensional extended objects [31, 36].

The question naturally arises whether every finite-dimensional Lie 2-algebra comes from a Lie 2-group. The answer is surprisingly subtle, as illustrated by a class of Lie 2-algebras coming from simple Lie algebras. Suppose G is a simply-connected compact simple Lie group G, and let $\mathfrak{g}$ be its Lie algebra. For any real number k, there is a Lie 2-algebra $\mathfrak{g}_k$ for which the space of objects is $\mathfrak{g}$, the space of endomorphisms of any object is $\mathbb{R}$, and the 'Jacobiator'

$$J_{x,y,z} \colon [[x,y],z] \xrightarrow{\sim} [x,[y,z]] + [[x,z],y]$$

is given by

$$J_{x,y,z} = k\,\nu(x,y,z)$$

where ν is as above. If we normalize the invariant inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ so that the de Rham cohomology class of the closed form $\nu/2\pi$ generates the third integral cohomology of G, then there is a 2-group G_k corresponding to $\mathfrak{g}_k$ in a certain sense explained below whenever k is an integer. The construction of this 2-group is very interesting, because it uses Chern–Simons theory in an essential way. However, for $k \neq 0$ there is no good way to make this 2-group into a Lie 2-group! The set of objects is naturally a smooth manifold, and so is the set of morphisms, and the group operations are smooth, but the associator

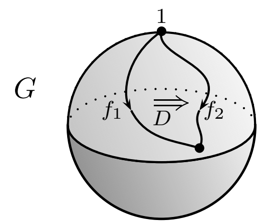
$$a_{x,y,z} \colon (xy)z \xrightarrow{\sim} x(yz)$$

cannot be made everywhere smooth, or even continuous.

It would be disappointing if such a fundamental Lie 2-algebra as $\mathfrak{g}_k$ failed to come from a Lie 2-group even when k was an integer. Here we resolve this dilemma by finding a Lie 2-algebra *equivalent* to $\mathfrak{g}_k$ that *does* come from a Lie 2-group — albeit an *infinite-dimensional* one.

The point is that the natural concept of 'sameness' for categories is a bit subtle: not isomorphism, but equivalence. Two categories are 'equivalent' if there are functors going back and forth between them that are inverses *up to natural isomorphism.* Categories that superficially look quite different can turn out to be equivalent. The same is true for 2-groups and Lie 2-algebras. Taking advantage of this, we show that while the finite-dimensional Lie 2-algebra $\mathfrak{g}_k$ has no corresponding Lie 2-group, it is equivalent to an infinite-dimensional Lie 2-algebra $\mathcal{P}_k\mathfrak{g}$ which comes from an infinite-dimensional Lie 2-group $\mathcal{P}_kG$.

The 2-group $\mathcal{P}_kG$ is easy to describe, in part because it is 'strict': all the usual group axioms hold as equations. The basic idea is easiest to understand using some geometry. Apart from some technical fine print, an object of $\mathcal{P}_kG$ is just a path in G starting at the identity. A morphism from the path f_1 to the path f_2 is an equivalence class of pairs (D,z) consisting of a disk D going from f_1 to f_2 together with a unit complex number z:



Given two such pairs (D_1, z_1) and (D_2, z_2), we can always find a 3-ball B whose boundary is $D_1 \cup D_2$, and we say the pairs are equivalent when

$$z_2/z_1 = e^{ik\int_B \nu}$$

where ν is the left-invariant closed 3-form on G given as above. Note that $\exp(ik\int_B \nu)$ is independent of the choice of B, because the integral of ν over any 3-sphere is 2π times an integer. There is an obvious way to compose morphisms in $\mathcal{P}_k G$, and the resulting category inherits a Lie 2-group structure from the Lie group structure of G.

The above description of $\mathcal{P}_k G$ is modeled after Murray's construction [176] of a gerbe from an integral closed 3-form on a manifold with a chosen basepoint. Indeed, $\mathcal{P}_k G$ is just another way of talking about the kth power of the canonical gerbe on G, and the 2-group structure on $\mathcal{P}_k G$ is a reflection of the fact that this gerbe is 'multiplicative' in the sense of Brylinski [181]. The 3-form $k\nu$, which plays the role of the Jacobiator in $\mathfrak{g}_k$, is the 3-curvature of a connection on this gerbe.

In most of this paper we take a slightly different viewpoint. Let $P_0 G$ be the space of smooth paths $f\colon [0, 2\pi] \to G$ that start at the identity of G. This becomes an infinite-dimensional Lie group under pointwise multiplication. The map $f \mapsto f(2\pi)$ is a homomorphism from $P_0 G$ to G whose kernel is precisely ΩG. For any $k \in \mathbb{Z}$, the loop group ΩG has a central extension

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \widehat{\Omega_k G} \xrightarrow{p} \Omega G \longrightarrow 1$$

which at the Lie algebra level is determined by the 2-cocycle $ik\omega$, with ω defined as above. This is called the 'level-k Kac–Moody central extension' of G. The infinite-dimensional Lie 2-group $\mathcal{P}_k G$ has $P_0 G$ as its group of objects, and given $f_1, f_2 \in P_0 G$, a morphism $\hat{\ell}\colon f_1 \to f_2$ is an element $\hat{\ell} \in \widehat{\Omega_k G}$ such that

$$f_2/f_1 = p(\hat{\ell}).$$

In this description, composition of morphisms in $\mathcal{P}_k G$ is multiplication in $\widehat{\Omega_k G}$, while again $\mathcal{P}_k G$ becomes a Lie 2-group using the Lie group structure of G.

To better understand the significance of the Lie 2-algebra $\mathfrak{g}_k$ and the 2-group G_k it is helpful to recall the classification of 2-groups and Lie 2-algebras. In [40] it is shown that Lie 2-algebras are classified up to equivalence by quadruples consisting of:

- a Lie algebra $\mathfrak{g}$,
- an abelian Lie algebra $\mathfrak{h}$,

- a representation ρ of $\mathfrak{g}$ on $\mathfrak{h}$,
- an element $[j] \in H^3(\mathfrak{g}, \mathfrak{h})$ of the Lie algebra cohomology of $\mathfrak{g}$.

Given a Lie 2-algebra $\mathfrak{c}$, we obtain this data by choosing a 'skeleton' $\mathfrak{c}_0$ of $\mathfrak{c}$: that is, an equivalent Lie 2-algebra in which any pair of isomorphic objects are equal. The objects in this skeleton form the Lie algebra $\mathfrak{g}$, while the endomorphisms of any object form the abelian Lie algebra $\mathfrak{h}$. The representation of $\mathfrak{g}$ on $\mathfrak{h}$ comes from the bracket in $\mathfrak{c}_0$, and the element $[j]$ comes from the Jacobiator.

Similarly, in [41] we give a proof of the already known fact that 2-groups are classified up to equivalence by quadruples consisting of:

- a group G,
- an abelian group H,
- an action α of G as automorphisms of H,
- an element $[a] \in H^3(G, H)$ of group cohomology of G.

Given a 2-group C, we obtain this data by choosing a skeleton C_0: that is, an equivalent 2-group in which any pair of isomorphic objects are equal. The objects in this skeleton form the group G, while the automorphisms of any object form the abelian group H. The action of G on H comes from conjugation in C_0, and the element $[a]$ comes from the associator.

These strikingly parallel classifications suggest that 2-groups should behave like Lie 2-algebras to the extent that group cohomology resembles Lie algebra cohomology. But this is where the subtleties begin!

Suppose G is a simply-connected compact simple Lie group, and let $\mathfrak{g}$ be its Lie algebra. If ρ is the trivial representation of $\mathfrak{g}$ on $\mathfrak{u}(1)$, we have

$$H^3(\mathfrak{g}, \mathfrak{u}(1)) \cong \mathbb{R}$$

because this cohomology group can be identified with the third de Rham cohomology group of G, which has the class $[\nu]$ as a basis. Thus, for any $k \in \mathbb{R}$ we obtain a skeletal Lie 2-algebra $\mathfrak{g}_k$ having $\mathfrak{g}$ as its Lie algebra of objects and $\mathfrak{u}(1)$ as the endomorphisms of any object, where the Jacobiator in $\mathfrak{g}_k$ is given by

$$J_{x,y,z} = k\nu(x, y, z).$$

To build a 2-group G_k analogous to this Lie 2-algebra $\mathfrak{g}_k$, we need to understand the relation between $H^3(G, \mathrm{U}(1))$ and $H^3(\mathfrak{g}, \mathfrak{u}(1))$. They are not isomorphic. However, $H^3(\mathfrak{g}, \mathfrak{u}(1))$ contains a lattice Λ consisting of the integer multiples of $[\nu]$. The papers of Chern–Simons [182] and Cheeger–Simons [183] construct an inclusion

$$\iota \colon \Lambda \hookrightarrow H^3(G, \mathrm{U}(1)).$$

Thus, when k is an integer, we can build a skeletal 2-group G_k having G as its group of objects, $\mathrm{U}(1)$ as the group of automorphisms of any object, the trivial action of G on $\mathrm{U}(1)$, and $[a] \in H^3(G, \mathrm{U}(1))$ given by $k\,\iota[\nu]$.

The question naturally arises whether G_k can be made into a Lie 2-group. The problem is that there is no continuous representative of the cohomology class $k\,\iota[\nu]$ unless $k = 0$. Thus, for k nonzero, we cannot make G_k into a Lie 2-group in any reasonable way. More precisely, we have this result [41]:

Theorem 10.1. *Let G be a simply-connected compact simple Lie group. Unless $k = 0$, there is no way to give the 2-group G_k the structure of a Lie 2-group for which the group G of objects and the group $\mathrm{U}(1)$ of endomorphisms of any object are given their usual topology.*

The goal of this paper is to sidestep this 'no-go theorem' by finding a Lie 2-algebra equivalent to $\mathfrak{g}_k$ which does come from an (infinite-dimensional) Lie group when $k \in \mathbb{Z}$. We show:

Theorem 10.2. *Let G be a simply-connected compact simple Lie group. For any $k \in \mathbb{Z}$, there is a Fréchet Lie 2-group $\mathcal{P}_k G$ whose Lie 2-algebra $\mathcal{P}_k \mathfrak{g}$ is equivalent to $\mathfrak{g}_k$.*

We also study the relation between $\mathcal{P}_k G$ and the topological group $\hat{G}$ obtained by killing the third homotopy group of G. When $G = \mathrm{Spin}(n)$, this topological group is famous under the name of $\mathrm{String}(n)$, since it plays a role in string theory [87, 20, 18]. More generally, any compact simple Lie group G has $\pi_3(G) = \mathbb{Z}$, but after killing $\pi_1(G)$ by passing to the universal cover of G, one can then kill $\pi_3(G)$ by passing to $\hat{G}$, which is defined as the homotopy fiber of the canonical map from G to the Eilenberg–Mac Lane space $K(\mathbb{Z}, 3)$. This specifies $\hat{G}$ up to homotopy, but there is still the interesting problem of finding nice geometrical models for $\hat{G}$.

Before presenting their solution to this problem, Stolz and Teichner [20] wrote: "To our best knowledge, there has yet not been found a canonical construction for $\mathrm{String}(n)$ which has reasonable 'size' and a geometric interpretation." Here we present another solution. There is a way to turn any topological 2-group C into a topological group $|C|$, which we explain in Section 10.4.2. Applying this to $\mathcal{P}_k G$ when $k = \pm 1$, we obtain $\hat{G}$:

Theorem 10.3. *Let G be a simply-connected compact simple Lie group. Then $|\mathcal{P}_k G|$ is an extension of G by a topological group that is homotopy equivalent to $K(\mathbb{Z}, 2)$. Moreover, $|\mathcal{P}_k G| \simeq \hat{G}$ when $k = \pm 1$.*

While this construction of $\hat{G}$ uses simplicial methods and is thus arguably less 'geometric' than that of Stolz and Teichner, it avoids their use of type III_1 von Neumann algebras, and has a simple relation to the Kac–Moody central extension of G.

10.2 2-Groups and 2-Algebras

We begin with a review of Lie 2-algebras and Lie 2-groups. More details can be found in our papers HDA5 [41] and HDA6 [40]. Our notation largely follows that of these papers, but the reader should be warned that here we denote the composite of morphisms $f\colon x \to y$ and $g\colon y \to z$ as $g \circ f\colon x \to z$.

10.2.1 Lie 2-algebras

The concept of 'Lie 2-algebra' blends together the notion of a Lie algebra with that of a category. Just as a Lie algebra has an underlying vector space, a Lie 2-algebra has an underlying 2-vector space: that is, a category where everything is *linear*. More precisely, a **2-vector space** L is a category for which:

- the set of objects $\mathrm{Ob}(L)$,

- the set of morphisms $\mathrm{Mor}(L)$

are both vector spaces, and:

- the maps $s, t\colon \mathrm{Mor}(L) \to \mathrm{Ob}(L)$ sending any morphism to its source and target,

- the map $i\colon \mathrm{Ob}(L) \to \mathrm{Mor}(L)$ sending any object to its identity morphism,

- the map $\circ$ sending any composable pair of morphisms to its composite

are all linear. As usual, we write a morphism as $f\colon x \to y$ when $s(f) = x$ and $t(f) = y$, and we often write $i(x)$ as 1_x.

To obtain a Lie 2-algebra, we begin with a 2-vector space and equip it with a bracket functor, which satisfies the Jacobi identity up to a natural isomorphism called the 'Jacobiator'. Then we require that the Jacobiator satisfy a new coherence law of its own: the 'Jacobiator identity'.

Definition 10.1. *A* **Lie 2-algebra** *consists of:*

- *a 2-vector space L*

equipped with:

- *a functor called the* **bracket**
$$[\cdot,\cdot]\colon L \times L \to L,$$
bilinear and skew-symmetric as a function of objects and morphisms,

- *a natural isomorphism called the* **Jacobiator***,*
$$J_{x,y,z}\colon [[x,y],z] \to [x,[y,z]] + [[x,z],y],$$
trilinear and antisymmetric as a function of the objects $x, y, z \in L$,

such that:

- *the* **Jacobiator identity** *holds: the following diagram commutes for all objects* $w, x, y, z \in L$*:*

$$
\begin{tikzcd}[column sep=tiny]
& {[[[w,x],y],z]} \arrow[dl, "{[J_{w,x,y},z]}"'] \arrow[dr, "{J_{[w,x],y,z}}"] & \\
{[[[w,y],x],z]+[[w,[x,y]],z]} \arrow[d, "{J_{[w,y],x,z}+J_{w,[x,y],z}}"'] & & {[[[w,x],z],y]+[[w,x],[y,z]]} \arrow[d, "{[J_{w,x,z},y]+1}"] \\
{\substack{[[[w,y],z],x]+[[w,y],[x,z]] \\ +[w,[[x,y],z]]+[[w,z],[x,y]]}} \arrow[d, "{[J_{w,y,z},x]+1}"'] & & {\substack{[[w,[x,z]],y] \\ +[[w,x],[y,z]]+[[[w,z],x],y]}} \arrow[d, "{J_{w,[x,z],y}+J_{[w,z],x,y}+J_{w,x,[y,z]}}"] \\
{\substack{[[[w,z],y],x]+[[w,[y,z]],x] \\ +[[w,y],[x,z]]+[w,[[x,y],z]]+[[w,z],[x,y]]}} \arrow[rr, "{[w,J_{x,y,z}]+1}"'] & & {\substack{[[[w,z],y],x]+[[w,z],[x,y]]+[[w,y],[x,z]] \\ +[w,[[x,z],y]]+[[w,[y,z]],x]+[w,[x,[y,z]]]}}
\end{tikzcd}
$$

A homomorphism between Lie 2-algebras is a linear functor preserving the bracket, but only up to a specified natural isomorphism satisfying a suitable coherence law. More precisely:

Definition 10.2. *Given Lie* 2*-algebras* L *and* L'*, a* **homomorphism** $F\colon L \to L'$ *consists of:*

- *a functor* F *from the underlying* 2*-vector space of* L *to that of* L'*, linear on objects and morphisms,*
- *a natural isomorphism*

$$F_2(x,y)\colon [F(x), F(y)] \to F[x,y],$$

bilinear and skew-symmetric as a function of the objects $x, y \in L$*,*

such that:

- *the following diagram commutes for all objects* $x, y, z \in L$*:*

$$
\begin{tikzcd}
{[F(x),[F(y),F(z)]]} \arrow[r, "{J_{F(x),F(y),F(z)}}"] \arrow[d, "{[1,F_2]}"'] & {[[F(x),F(y)],F(z)]+[F(y),[F(x),F(z)]]} \arrow[d, "{[F_2,1]+[1,F_2]}"] \\
{[F(x),F[y,z]]} \arrow[d, "F_2"'] & {[F[x,y],F(z)]+[F(y),F[x,z]]} \arrow[d, "F_2+F_2"] \\
{F[x,[y,z]]} \arrow[r, "{F(J_{x,y,z})}"] & {F[[x,y],z]+F[y,[x,z]]}
\end{tikzcd}
$$

Here and elsewhere we omit the arguments of natural transformations such as F_2 and G_2 when these are obvious from context.

Similarly, a '2-homomorphism' is a linear natural isomorphism that is compatible with the bracket structure:

Definition 10.3. *Let $F, G \colon L \to L'$ be Lie 2-algebra homomorphisms. A* **2-homomorphism** *$\theta \colon F \Longrightarrow G$ is a natural transformation*

$$\theta_x \colon F(x) \to G(x),$$

linear as a function of the object $x \in L$, such that the following diagram commutes for all $x, y \in L$:

$$\begin{tikzcd}
{[F(x), F(y)]} \arrow[r, "F_2"] \arrow[d, "{[\theta_x, \theta_y]}"'] & F[x,y] \arrow[d, "\theta_{[x,y]}"] \\
{[G(x), G(y)]} \arrow[r, "G_2"] & G[x,y]
\end{tikzcd}$$

In HDA6 we showed:

Proposition 10.1. *There is a strict 2-category* **Lie2Alg** *with Lie 2-algebras as objects, homomorphisms between these as morphisms, and 2-homomorphisms between those as 2-morphisms.*

10.2.2 L_∞-algebras

Just as the concept of Lie 2-algebra blends the notions of Lie algebra and category, the concept of 'L_∞-algebra' blends the notions of Lie algebra and chain complex. More precisely, an L_∞-algebra is a chain complex equipped with a bilinear skew-symmetric bracket operation that satisfies the Jacobi identity up to a chain homotopy, which in turn satisfies a law of its own up to chain homotopy, and so on *ad infinitum*. In fact, L_∞-algebras were defined long before Lie 2-algebras, going back to a 1985 paper by Schlessinger and Stasheff [184]. They are also called 'strongly homotopy Lie algebras', or 'sh Lie algebras' for short.

Our conventions regarding L_∞-algebras follow those of Lada and Markl [185]. In particular, for graded objects $x_1, \ldots, x_n$ and a permutation $\sigma \in S_n$ we define the **Koszul sign** $\epsilon(\sigma)$ by the equation

$$x_1 \wedge \cdots \wedge x_n = \epsilon(\sigma)\, x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(n)},$$

which must be satisfied in the free graded-commutative algebra on $x_1, \ldots, x_n$. Furthermore, we define

$$\chi(\sigma) = \mathrm{sgn}(\sigma)\, \epsilon(\sigma; x_1, \ldots, x_n).$$

Thus, $\chi(\sigma)$ takes into account the sign of the permutation in S_n as well as the Koszul sign. Finally, if n is a natural number and $1 \leq j \leq n-1$ we say that $\sigma \in S_n$ is an $(j, n-j)$**-unshuffle** if

$$\sigma(1) \leq \sigma(2) \leq \cdots \leq \sigma(j) \quad \text{and} \quad \sigma(j+1) \leq \sigma(j+2) \leq \cdots \leq \sigma(n).$$

Readers familiar with shuffles will recognize unshuffles as their inverses.

Definition 10.4. *An* $\boldsymbol{L_\infty}$**-algebra** *is a graded vector space* V *equipped with a system* $\{l_k | 1 \leq k < \infty\}$ *of linear maps* $l_k \colon V^{\otimes k} \to V$ *with* $\deg(l_k) = k-2$ *which are totally antisymmetric in the sense that*

$$l_k(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) = \chi(\sigma) l_k(x_1, \ldots, x_n) \tag{10.1}$$

for all $\sigma \in S_n$ *and* $x_1, \ldots, x_n \in V$*, and, moreover, the following generalized form of the Jacobi identity holds for* $0 \leq n < \infty$ *:*

$$\sum_{i+j=n+1} \sum_{\sigma} \chi(\sigma)(-1)^{i(j-1)} l_j(l_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}) = 0, \tag{10.2}$$

where the summation is taken over all $(i, n-i)$*-unshuffles with* $i \geq 1$*.*

In this definition the map l_1 makes V into a chain complex, since this map has degree -1 and Equation (10.2) says its square is zero. In what follows, we denote l_1 as d. The map l_2 resembles a Lie bracket, since it is skew-symmetric in the graded sense by Equation (10.1). The higher l_k maps are related to the Jacobiator and the Jacobiator identity.

To make this more precise, we make the following definition:

Definition 10.5. *A* $\boldsymbol{k}$**-term** $\boldsymbol{L_\infty}$**-algebra** *is an* L_∞*-algebra* V *with* $V_n = 0$ *for* $n \geq k$*.*

A 1-term L_∞-algebra is simply an ordinary Lie algebra, where $l_3 = 0$ gives the Jacobi identity. However, in a 2-term L_∞-algebra, we no longer have $l_3 = 0$. Instead, Equation (10.2) says that the Jacobi identity for $x, y, z \in V_0$ holds up to a term of the form $dl_3(x, y, z)$. We do, however, have $l_4 = 0$, which provides us with the coherence law that l_3 must satisfy. It follows that a 2-term L_∞-algebra consists of:

- vector spaces V_0 and V_1,
- a linear map $d \colon V_1 \to V_0$,
- bilinear maps $l_2 \colon V_i \times V_j \to V_{i+j}$, where $0 \leq i + j \leq 1$,
- a trilinear map $l_3 \colon V_0 \times V_0 \times V_0 \to V_1$

satisfying a list of equations coming from Equations (10.1) and (10.2) and the fact that $l_4 = 0$. This list can be found in HDA6, but we will not need it here.

In fact, 2-vector spaces are equivalent to 2-term chain complexes of vector spaces: that is, chain complexes of the form

$$V_1 \xrightarrow{d} V_0.$$

To obtain such a chain complex from a 2-vector space L, we let V_0 be the space of objects of L. However, V_1 is not the space of morphisms. Instead, we define the **arrow part** $\vec{f}$ of a morphism $f \colon x \to y$ by

$$\vec{f} = f - i(s(f)),$$

and let V_1 be the space of these arrow parts. The map $d\colon V_1 \to V_0$ is then just the target map $t\colon \mathrm{Mor}(L) \to \mathrm{Ob}(L)$ restricted to $V_1 \subseteq \mathrm{Mor}(L)$.

To understand this construction a bit better, note that given any morphism $f\colon x \to y$, its arrow part is a morphism $\vec{f}\colon 0 \to y - x$. Thus, taking the arrow part has the effect of 'translating f to the origin'. We can always recover any morphism from its source together with its arrow part, since $f = \vec{f} + i(s(f))$. It follows that any morphism $f\colon x \to y$ can be identified with the ordered pair $(x, \vec{f})$ consisting of its source and arrow part. So, we have $\mathrm{Mor}(L) \cong V_0 \oplus V_1$.

We can actually recover the whole 2-vector space structure of L from just the chain complex $d\colon V_1 \to V_0$. To do this, we take:

$$\begin{aligned} \mathrm{Ob}(L) &= V_0 \\ \mathrm{Mor}(L) &= V_0 \oplus V_1, \end{aligned}$$

with source, target and identity-assigning maps defined by:

$$\begin{aligned} s(x, \vec{f}) &= x \\ t(x, \vec{f}) &= x + d\vec{f} \\ i(x) &= (x, 0) \end{aligned}$$

and with the composite of $f\colon x \to y$ and $g\colon y \to z$ defined by:

$$g \circ f = (x, \vec{f} + \vec{g}).$$

So, 2-vector spaces are equivalent to 2-term chain complexes.

Given this, it should not be surprising that Lie 2-algebras are equivalent to 2-term L_∞-algebras. Since we make frequent use of this fact in the calculations to come, we recall the details here.

Suppose V is a 2-term L_∞-algebra. We obtain a 2-vector space L from the underlying chain complex of V as above. We continue by equipping L with additional structure that makes it a Lie 2-algebra. It is sufficient to define the bracket functor $[\cdot,\cdot]\colon L \times L \to L$ on a pair of objects and on a pair of morphisms where one is an identity morphism. So, we set:

$$\begin{aligned} [x, y] &= l_2(x, y), \\ [1_z, f] &= (l_2(z, x), l_2(z, \vec{f})), \\ [f, 1_z] &= (l_2(x, z), l_2(\vec{f}, z)), \end{aligned}$$

where $f\colon x \to y$ is a morphism in L and z is an object. Finally, we define the Jacobiator for L in terms of its source and arrow part as follows:

$$J_{x,y,z} = ([[x, y], z], l_3(x, y, z)).$$

For a proof that L defined this way is actually a Lie 2-algebra, see HDA6.

In our calculations we shall often describe Lie 2-algebra homomorphisms as homomorphisms between the corresponding 2-term L_∞-algebras:

Definition 10.6. *Let V and V' be 2-term L_∞-algebras. An* $\boldsymbol{L_\infty}$**-homomorphism** $\phi\colon V \to V'$ *consists of:*

- *a chain map $\phi\colon V \to V'$ consisting of linear maps $\phi_0\colon V_0 \to V_0'$ and $\phi_1\colon V_1 \to V_1'$,*
- *a skew-symmetric bilinear map $\phi_2\colon V_0 \times V_0 \to V_1'$,*

such that the following equations hold for all $x, y, z \in V_0$ and $h \in V_1$:

$$d(\phi_2(x,y)) = \phi_0(l_2(x,y)) - l_2(\phi_0(x), \phi_0(y)) \tag{10.3}$$

$$\phi_2(x, dh) = \phi_1(l_2(x,h)) - l_2(\phi_0(x), \phi_1(h)) \tag{10.4}$$

$$\begin{array}{l} l_3(\phi_0(x), \phi_0(y), \phi_0(z)) - \phi_1(l_3(x,y,z)) = \\ \phi_2(x, l_2(y,z)) + \phi_2(y, l_2(z,x)) + \phi_2(z, l_2(x,y)) + \\ l_2(\phi_0(x), \phi_2(y,z)) + l_2(\phi_0(y), \phi_2(z,x)) + l_2(\phi_0(z), \phi_2(x,y)) \end{array} \tag{10.5}$$

Equations (10.3) and (10.4) say that ϕ_2 defines a chain homotopy from $l_2(\phi(\cdot), \phi(\cdot))$ to $\phi(l_2(\cdot,\cdot))$, where these are regarded as chain maps from $V \otimes V$ to V'. Equation (10.5) is just a chain complex version of the commutative diagram in Definition 10.2.

Without providing too many details, let us sketch how to obtain the Lie 2-algebra homomorphism F corresponding to a given L_∞-homomorphism $\phi\colon V \to V'$. We define the chain map $F\colon L \to L'$ in terms of ϕ using the fact that objects of a 2-vector space are 0-chains in the corresponding chain complex, while morphisms are pairs consisting of a 0-chain and a 1-chain. To make F into a Lie 2-algebra homomorphism we must equip it with a skew-symmetric bilinear natural transformation F_2 satisfying the conditions in Definition 10.2. We do this using the skew-symmetric bilinear map $\phi_2\colon V_0 \times V_0 \to V_1'$. In terms of its source and arrow parts, we let

$$F_2(x,y) = (l_2(\phi_0(x), \phi_0(y)), \phi_2(x,y)).$$

We should also know how to compose L_∞-homomorphisms. We compose a pair of L_∞-homomorphisms $\phi\colon V \to V'$ and $\psi\colon V' \to V''$ by letting the chain map $\psi \circ \phi\colon V \to V''$ be the usual composite:

$$V \xrightarrow{\phi} V' \xrightarrow{\psi} V''$$

while defining $(\psi \circ \phi)_2$ as follows:

$$(\psi \circ \phi)_2(x,y) = \psi_2(\phi_0(x), \phi_0(y)) + \psi_1(\phi_2(x,y)). \tag{10.6}$$

This is just a chain complex version of how we compose homomorphisms between Lie 2-algebras. Note that the identity homomorphism $1_V\colon V \to V$ has the identity chain map as its underlying map, together with $(1_V)_2 = 0$.

We also have '2-homomorphisms' between homomorphisms:

Definition 10.7. *Let V and V' be 2-term L_∞-algebras and let $\phi, \psi \colon V \to V'$ be L_∞-homomorphisms. An* ***L_∞*-2-homomorphism** *$\tau \colon \phi \Longrightarrow \psi$ is a chain homotopy τ from ϕ to ψ such that the following equation holds for all $x, y \in V_0$:*

$$\phi_2(x, y) - \psi_2(x, y) = l_2(\phi_0(x), \tau(y)) + l_2(\tau(x), \psi_0(y)) - \tau(l_2(x, y)) \tag{10.7}$$

Given an L_∞-2-homomorphism $\tau \colon \phi \Longrightarrow \psi$ between L_∞-homomorphisms $\phi, \psi \colon V \to V'$, there is a corresponding Lie 2-algebra 2-homomorphism θ whose source and arrow part are:

$$\theta(x) = (\phi_0(x), \tau(x))$$

for any object x. Checking that this really is a Lie 2-algebra 2-homomorphism is routine. In particular, Equation (10.7) is just a chain complex version of the commutative diagram in the Definition 10.3.

In HDA6, we showed:

Proposition 10.2. *There is a strict 2-category* $\mathbf{2TermL_\infty}$ *with 2-term L_∞-algebras as objects, L_∞-homomorphisms as morphisms, and L_∞-2-homomorphisms as 2-morphisms.*

Using the equivalence between 2-vector spaces and 2-term chain complexes, we established the equivalence between Lie 2-algebras and 2-term L_∞-algebras:

Theorem 10.4. *The 2-categories* Lie2Alg *and* $\mathrm{2TermL_\infty}$ *are 2-equivalent.*

We use this result extensively in Section 10.5. Instead of working in Lie2Alg, we do calculations in $\mathrm{2TermL_\infty}$. The reason is that defining Lie 2-algebra homomorphisms and 2-homomorphisms would require specifying both source and arrow parts of morphisms, while defining the corresponding L_∞-morphisms and 2-morphisms only requires us to specify the arrow parts. Manipulating the arrow parts rather than the full-fledged morphisms leads to less complicated computations.

10.2.3 The Lie 2-Algebra $\mathfrak{g}_k$

Another benefit of the equivalence between Lie 2-algebras and L_∞-algebras is that it gives some important examples of Lie 2-algebras. Instead of thinking of a Lie 2-algebra as a category equipped with extra structure, we may work with a 2-term chain complex endowed with the structure described in Definition 10.4. This is especially simple when the differential d vanishes. Thanks to the formula

$$d\vec{f} = t(f) - s(f),$$

this implies that the source of any morphism in the Lie 2-algebra equals its target. In other words, the corresponding Lie 2-algebra is 'skeletal':

Definition 10.8. *A category is* **skeletal** *if isomorphic objects are always equal.*

Every category is equivalent to a skeletal one formed by choosing one representative of each isomorphism class of objects [89]. As shown in HDA6, the same sort of thing is true in the context of Lie 2-algebras:

Proposition 10.3. *Every Lie* 2*-algebra is equivalent, as an object of* Lie2Alg*, to a skeletal one.*

This result helps us classify Lie 2-algebras up to equivalence. We begin by reminding the reader of the relationship between L_∞-algebras and Lie algebra cohomology described in HDA6:

Theorem 10.5. *There is a one-to-one correspondence between isomorphism classes of L_∞-algebras consisting of only two nonzero terms V_0 and V_n with $d = 0$, and isomorphism classes of quadruples $(\mathfrak{g}, V, \rho, [l_{n+2}])$ where $\mathfrak{g}$ is a Lie algebra, V is a vector space, ρ is a representation of $\mathfrak{g}$ on V, and $[l_{n+2}]$ is an element of the Lie algebra cohomology group $H^{n+2}(\mathfrak{g}, V)$.*

Here the representation ρ comes from $\ell_2 \colon V_0 \times V_n \to V_n$.

Because L_∞-algebras are equivalent to Lie 2-algebras, which all have equivalent skeletal versions, Theorem 10.5 implies:

Corollary 10.6. *Up to equivalence, Lie 2-algebras are classified by isomorphism classes of quadruples $(\mathfrak{g}, \rho, V, [\ell_3])$ where:*

- *$\mathfrak{g}$ is a Lie algebra,*
- *V is a vector space,*
- *ρ is a representation of $\mathfrak{g}$ on V,*
- *$[\ell_3]$ is an element of $H^3(\mathfrak{g}, V)$.*

This classification of Lie 2-algebras is just another way of stating the result mentioned in the Introduction. And, as mentioned there, this classification lets us construct a 1-parameter family of Lie 2-algebras $\mathfrak{g}_k$ for any simple real Lie algebra $\mathfrak{g}$:

Example 10.7. *Suppose $\mathfrak{g}$ is a simple real Lie algebra and $k \in \mathbb{R}$. Then there is a skeletal Lie 2-algebra $\mathfrak{g}_k$ given by taking $V_0 = \mathfrak{g}$, $V_1 = \mathbb{R}$, ρ the trivial representation, and $l_3(x, y, z) = k\langle x, [y, z]\rangle$.*

Here $\langle \cdot, \cdot \rangle$ is a suitably rescaled version of the Killing form $\mathrm{tr}(\mathrm{ad}(\cdot)\mathrm{ad}(\cdot))$. The precise rescaling factor will only become important in Section 10.3.1. The equation saying that l_3 is a 3-cocycle is equivalent to the equation saying that the left-invariant 3-form ν on G with $\nu(x, y, z) = \langle x, [y, z]\rangle$ is *closed.*

10.2.4 The Lie 2-Algebra of a Fréchet Lie 2-Group

Just as Lie groups have Lie algebras, 'strict Lie 2-groups' have 'strict Lie 2-algebras'. Strict Lie 2-groups and Lie 2-algebras are categorified versions of Lie groups and Lie algebras in which all laws hold 'on the nose' as equations, rather than up to isomorphism. All the Lie 2-groups discussed in this paper are strict. However, most of them are infinite-dimensional 'Fréchet' Lie 2-groups.

Since the concept of a Fréchet Lie group is easy to explain but perhaps not familiar to all readers, we begin by recalling this. For more details we refer the interested reader to the survey article by Milnor [186], or Pressley and Segal's book on loop groups [172].

A **Fréchet space** is a vector space equipped with a topology given by a countable family of seminorms $\|\cdot\|_n$, or equivalently by the metric

$$d(x, y) = \sum_n 2^{-n} \frac{\|x - y\|_n}{\|x - y\|_n + 1} ,$$

where we require that this metric be complete. A classic example is the space of smooth maps from the interval or circle to a finite-dimensional normed vector space, where $\|f\|_n$ is the supremum of the norm of the nth derivative of f. In particular, the space of smooth paths or loops in a finite-dimensional simple Lie algebra is a Fréchet space. This is the sort of example we shall need.

The theory of manifolds generalizes from the finite-dimensional case to the infinite-dimensional case by replacing $\mathbb{R}^n$ with a Fréchet space [187]. In particular, there is a concept of the 'Fréchet derivative' of a map between Fréchet spaces, and higher derivatives of such maps can also be defined. If V, W are Fréchet spaces and $U \subset V$ is an open set, a map $\phi\colon U \to W$ is called **smooth** if its nth derivative exists for all n. A **Fréchet manifold** modeled on the Fréchet space V is a paracompact Hausdorff space M that can be covered with open sets U_α equipped with homeomorphisms $\phi_\alpha\colon U_\alpha \to V$ called **charts** such that the maps $\phi_\alpha \circ \phi_\beta^{-1}$ are smooth where defined. In particular, the space of smooth paths or loops in a compact simple Lie group G is naturally a Fréchet manifold modeled on the Fréchet space of smooth paths or loops in the Lie algebra $\mathfrak{g}$.

A map between Fréchet manifolds is **smooth** if composing it with charts and their inverses in the usual way, we get functions between Fréchet spaces that are smooth where defined. A **Fréchet Lie group** is a Fréchet manifold G such that the multiplication map $m\colon G \times G \to G$ and the inverse map $\mathrm{inv}\colon G \to G$ are smooth. A **homomorphism** of Fréchet Lie groups is a group homomorphism that is also smooth.

Finally:

Definition 10.9. *A* **strict Fréchet Lie 2-group** C *is a category such that:*

- *the set of objects* $\mathrm{Ob}(C)$ *and*
- *the set of morphisms* $\mathrm{Mor}(C)$

are both Fréchet Lie groups, and:

- *the maps* $s, t\colon \mathrm{Mor}(C) \to \mathrm{Ob}(L)$ *sending any morphism to its source and target,*

- *the map* $i\colon \mathrm{Ob}(C) \to \mathrm{Mor}(C)$ *sending any object to its identity morphism,*
- *the map* $\circ\colon \mathrm{Mor}(C) \times_{\mathrm{Ob}(C)} \mathrm{Mor}(C) \to \mathrm{Mor}(C)$ *sending any composable pair of morphisms to its composite*

are all Fréchet Lie group homomorphisms.

Here $\mathrm{Mor}(C) \times_{\mathrm{Ob}(C)} \mathrm{Mor}(C)$ is the set of composable pairs of morphisms, which we require to be a Fréchet Lie group.

Just as for ordinary Lie groups, taking the tangent space at the identity of a Fréchet Lie group gives a Lie algebra. Using this, it is not hard to see that strict Fréchet Lie 2-groups give rise to Lie 2-algebras. These Lie 2-algebras are actually 'strict':

Definition 10.10. *A Lie 2-algebra is* **strict** *if its Jacobiator is the identity.*

This means that the map l_3 vanishes in the corresponding L_∞-algebra. Alternatively:

Proposition 10.4. *A strict Lie 2-algebra is the same as a 2-vector space L such that:*

- $\mathrm{Ob}(L)$ *is equipped with the structure of a Lie algebra,*
- $\mathrm{Mor}(L)$ *is equipped with the structure of a Lie algebra,*

and:

- *the source and target maps* $s, t\colon \mathrm{Mor}(L) \to \mathrm{Ob}(L)$*,*
- *the identity-assigning map* $i\colon \mathrm{Ob}(L) \to \mathrm{Mor}(L)$*, and*
- *the composition map* $\circ\colon \mathrm{Mor}(L) \times_{\mathrm{Ob}(L)} \mathrm{Mor}(L) \to \mathrm{Mor}(L)$

are Lie algebra homomorphisms.

Proof. - A straightforward verification; see also HDA6. □

Proposition 10.5. *Given a strict Fréchet Lie* 2*-group* C*, there is a strict Lie* 2*-algebra* $\mathfrak{c}$ *for which:*

- $\mathrm{Ob}(\mathfrak{c})$ *is the Lie algebra of the Fréchet Lie group* $\mathrm{Ob}(C)$*,*
- $\mathrm{Mor}(\mathfrak{c})$ *is the Lie algebra of the Fréchet Lie group* $\mathrm{Mor}(C)$*,*

and the maps:

- $s, t\colon \mathrm{Mor}(\mathfrak{c}) \to \mathrm{Ob}(\mathfrak{c})$*,*
- $i\colon \mathrm{Ob}(\mathfrak{c}) \to \mathrm{Mor}(\mathfrak{c})$*, and*
- $\circ\colon \mathrm{Mor}(\mathfrak{c}) \times_{\mathrm{Ob}(\mathfrak{c})} \mathrm{Mor}(\mathfrak{c}) \to \mathrm{Mor}(\mathfrak{c})$

are the differentials of the corresponding maps for C.

Proof. This is a generalization of a result in HDA6 for ordinary Lie 2-groups, which is straightforward to show directly. □

In what follows all Fréchet Lie 2-groups are strict, so we omit the term 'strict'.

10.3 Loop Groups

Next we give a brief review of loop groups and their central extensions. More details can be found in the canonical text on the subject, written by Pressley and Segal [172].

10.3.1 Definitions and Basic Properties

Let G be a simply-connected compact simple Lie group. We shall be interested in the **loop group** ΩG consisting of all smooth maps from $[0, 2\pi]$ to G with $f(0) = f(2\pi) = 1$. We make ΩG into a group by pointwise multiplication of loops: $(fg)(\theta) = f(\theta)g(\theta)$. Equipped with its C^∞ topology, ΩG naturally becomes an infinite-dimensional Fréchet manifold. In fact ΩG is a Fréchet Lie group, as defined in Section 10.2.4.

As remarked by Pressley and Segal, the behaviour of the group ΩG is "untypical in its simplicity," since it turns out to behave remarkably like a compact Lie group. For example, it has an exponential map that is locally one-to-one and onto, and it has a well-understood highest weight theory of representations. One striking difference between ΩG and G, though, is the existence of nontrivial central extensions of ΩG by the circle $\mathrm{U}(1)$:

$$1 \to \mathrm{U}(1) \to \widehat{\Omega G} \xrightarrow{p} \Omega G \to 1. \tag{10.8}$$

It is important to understand that these extensions are nontrivial, not merely in that they are classified by a nonzero 2-cocycle, but also *topologically.* In other words, $\widehat{\Omega G}$ is a nontrivial principal $\mathrm{U}(1)$-bundle over ΩG with the property that $\widehat{\Omega G}$ is a Fréchet Lie group, and $\mathrm{U}(1)$ sits inside $\widehat{\Omega G}$ as a central subgroup in such a way that the quotient $\widehat{\Omega G}/\mathrm{U}(1)$ can be identified with ΩG. Perhaps the best analogy is with the double cover of $\mathrm{SO}(3)$: there $\mathrm{SU}(2)$ fibers over $\mathrm{SO}(3)$ as a 2-sheeted covering and $\mathrm{SU}(2)$ is not homeomorphic to $SO(3) \times \mathbb{Z}/2\mathbb{Z}$. $\widehat{\Omega G}$ is called the **Kac–Moody group**.

Associated to the central extension (10.8) there is a central extension of Lie algebras:

$$0 \to \mathfrak{u}(1) \to \widehat{\Omega \mathfrak{g}} \xrightarrow{p} \Omega \mathfrak{g} \to 0 \tag{10.9}$$

Here $\Omega\mathfrak{g}$ is the Lie algebra of ΩG, consisting of all smooth maps $f\colon S^1 \to \mathfrak{g}$ such that $f(0) = 0$. The bracket operation on $\Omega\mathfrak{g}$ is given by the pointwise bracket of functions: thus $[f, g](\theta) = [f(\theta), g(\theta)]$ if $f, g \in \Omega\mathfrak{g}$. $\widehat{\Omega\mathfrak{g}}$ is the simplest example of an affine Lie algebra.

The Lie algebra extension (10.9) is simpler than the group extension (10.8) in that it is determined up to isomorphism by a Lie algebra 2-cocycle $\omega(f, g)$, i.e. a skew bilinear map $\omega\colon \Omega\mathfrak{g} \times \Omega\mathfrak{g} \to \mathbb{R}$ satisfying the **2-cocycle condition**

$$\omega([f, g], h) + \omega([g, h], f) + \omega([h, f], g) = 0. \tag{10.10}$$

For G as above we may assume the cocycle ω equal, up to a scalar multiple, to the **Kac–Moody 2-cocycle**

$$\omega(f,g) = 2\int_0^{2\pi} \langle f(\theta), g'(\theta)\rangle \, d\theta \tag{10.11}$$

Here $\langle\cdot,\cdot\rangle$ is an invariant symmetric bilinear form on $\mathfrak{g}$. Thus, as a vector space $\widehat{\Omega\mathfrak{g}}$ is isomorphic to $\Omega\mathfrak{g}\oplus\mathbb{R}$, but the bracket is given by

$$[(f,\alpha),(g,\beta)] = ([f,g],\omega(f,g))$$

Since ω is a skew form on $\Omega\mathfrak{g}$, it defines a left-invariant 2-form ω on ΩG. The cocycle condition, Equation (10.10), says precisely that ω is closed. We quote the following theorem from Pressley and Segal, slightly corrected:

Theorem 10.8. *Suppose G is a simply-connected compact simple Lie group. Then:*

1. *The central extension of Lie algebras*

$$0 \to \mathfrak{u}(1) \to \widehat{\Omega\mathfrak{g}} \to \Omega\mathfrak{g} \to 0$$

defined by the cocycle ω above corresponds to a central extension of Fréchet Lie groups

$$1 \to \mathrm{U}(1) \to \widehat{\Omega G} \to \Omega G \to 1$$

in the sense that $i\omega$ is the curvature of a left-invariant connection on the principal $\mathrm{U}(1)$-bundle $\widehat{\Omega G}$ iff the 2-form $\omega/2\pi$ on ΩG has integral periods.

2. *The 2-form $\omega/2\pi$ has integral periods iff the invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ satisfies this integrality condition: $\langle h_\theta, h_\theta\rangle \in \frac{1}{2\pi}\mathbb{Z}$ for the coroot h_θ associated to the highest root θ of G.*

Since G is simple, all invariant symmetric bilinear forms on its Lie algebra are proportional, so there is a unique invariant inner product $(\cdot,\cdot)$ with $(h_\theta,h_\theta)=2$. Pressley and Segal [172] call this inner product the **basic inner product** on $\mathfrak{g}$. In what follows, we always use $\langle\cdot,\cdot\rangle$ to stand for this basic inner product divided by 4π. This is the smallest inner product to satisfy the integrality condition in the above theorem.

More generally, for any integer k, the inner product $k\langle\cdot,\cdot\rangle$ satisfies the integrality condition in Theorem 10.8. It thus gives rise to a central extension

$$1 \to \mathrm{U}(1) \to \widehat{\Omega_k G} \to \Omega G \to 1$$

of ΩG. The integer k is called the **level** of the central extension $\widehat{\Omega_k G}$.

10.3.2 The Kac–Moody group $\widehat{\Omega_k G}$

In this section we begin by recalling an explicit construction of $\widehat{\Omega_k G}$ due to Murray and Stevenson [87], inspired by the work of Mickelsson [188]. We then use this to prove a result, Proposition 10.6, that will be crucial for constructing the 2-group $\mathcal{P}_k G$.

First, suppose that $\mathcal{G}$ is any Fréchet Lie group. Let $P_0\mathcal{G}$ denote the space of smooth based paths in $\mathcal{G}$:

$$P_0\mathcal{G} = \{f \in C^\infty([0,2\pi],\mathcal{G})\colon\ f(0) = 1\}$$

$P_0\mathcal{G}$ is a Fréchet Lie group under pointwise multiplication of paths, whose Lie algebra is

$$P_0 L = \{f \in C^\infty([0,2\pi],L)\colon\ f(0) = 0\}$$

where L is the Lie algebra of $\mathcal{G}$. Furthermore, the map $\pi\colon P_0\mathcal{G} \to \mathcal{G}$ which evaluates a path at its endpoint is a homomorphism of Fréchet Lie groups. The kernel of π is equal to

$$\Omega\mathcal{G} = \{f \in C^\infty([0,2\pi],\mathcal{G})\colon\ f(0) = f(1) = 1\,\}$$

Thus, $\Omega\mathcal{G}$ is a normal subgroup of $P_0\mathcal{G}$. Note that we are defining $\Omega\mathcal{G}$ in a somewhat nonstandard way, since its elements can be thought of as loops $f\colon S^1 \to \mathcal{G}$ that are smooth everywhere except at the basepoint, where both left and right derivatives exist to all orders, but need not agree. However, we need this for the sequence

$$1 \longrightarrow \Omega\mathcal{G} \longrightarrow P_0\mathcal{G} \xrightarrow{\pi} \mathcal{G} \longrightarrow 1$$

to be exact, and our $\Omega\mathcal{G}$ is homotopy equivalent to the standard one.

At present we are most interested in the case where $\mathcal{G} = \Omega G$. Then a point in $P_0\mathcal{G}$ gives a map $f\colon [0,2\pi]\times S^1 \to G$ with $f(0,\theta) = 1$ for all $\theta \in S^1$, $f(t,0) = 1$ for all $t \in [0,2\pi]$. It is an easy calculation [87] to show that the map $\kappa\colon P_0\Omega G \times P_0\Omega G \to \mathrm{U}(1)$ defined by

$$\kappa(f,g) = \exp\left(2ik\int_0^{2\pi}\int_0^{2\pi} \langle f(t)^{-1}f'(t), g'(\theta)g(\theta)^{-1}\rangle\, d\theta\, dt\right) \tag{10.12}$$

is a group 2-cocycle. This 2-cocycle κ makes $P_0\Omega G \times \mathrm{U}(1)$ into a group with the following product:

$$(f_1,z_1)\cdot(f_2,z_2) = (f_1 f_2, z_1 z_2\, \kappa(f_1,f_2))\,.$$

Let N be the subset of $P_0\Omega G \times \mathrm{U}(1)$ consisting of pairs (γ,z) such that $\gamma\colon [0,2\pi] \to \Omega G$ is a loop based at $1 \in \Omega G$ and

$$z = \exp\left(-ik\int_{D_\gamma}\omega\right)$$

where D_γ is any disk in ΩG with γ as its boundary. It is easy to check that N is a normal subgroup of the group $P_0\Omega G \times \mathrm{U}(1)$ with the product defined as above. To construct $\widehat{\Omega_k G}$ we form the quotient group $(P_0\Omega G \times \mathrm{U}(1))/N$. In [87] it is shown that the resulting central extension is isomorphic to the central extension of ΩG at level k. So we have the commutative diagram

$$\begin{tikzcd}
P_0\Omega G \times \mathrm{U}(1) \arrow[r] \arrow[d] & \widehat{\Omega_k G} \arrow[d] \\
P_0\Omega G \arrow[r, "\pi"] & \Omega G
\end{tikzcd} \tag{10.13}$$

where the horizontal maps are quotient maps, the upper horizontal map corresponding to the normal subgroup N, and the lower horizontal map corresponding to the normal subgroup $\Omega^2 G$ of $P_0\Omega G$.

Notice that the group of based paths P_0G acts on ΩG by conjugation. The next proposition shows that this action lifts to an action on $\widehat{\Omega_k G}$:

Proposition 10.6. *The action of P_0G on ΩG by conjugation lifts to a smooth action α of P_0G on $\widehat{\Omega_k G}$, whose differential gives an action $d\alpha$ of the Lie algebra $P_0\mathfrak{g}$ on the Lie algebra $\widehat{\Omega_k \mathfrak{g}}$ with*

$$d\alpha(p)(\ell, c) = \big([p,\ell],\ 2k\int_0^{2\pi} \langle p(\theta), \ell'(\theta)\rangle\, d\theta\,\big).$$

for all $p \in P_0\mathfrak{g}$ and all $(\ell, c) \in \Omega\mathfrak{g} \oplus \mathbb{R} \cong \widehat{\Omega_k \mathfrak{g}}$.

Proof. To construct α it suffices to construct a smooth action of P_0G on $P_0\Omega G \times \mathrm{U}(1)$ that preserves the product on this group and also preserves the normal subgroup N. Let $p\colon [0, 2\pi] \to G$ be an element of P_0G, so that $p(0) = 1$. Define the action of p on a point $(f, z) \in P_0\Omega G \times \mathrm{U}(1)$ to be

$$p \cdot (f, z) = \big(pfp^{-1},\ z\exp(ik\int_0^{2\pi} \beta_p(f(t)^{-1}f'(t))\, dt)\ \big)$$

where β_p is the left-invariant 1-form on ΩG corresponding to the linear map $\beta_p\colon \Omega\mathfrak{g} \to \mathbb{R}$ given by:

$$\beta_p(\xi) = -2\int_0^{2\pi} \langle \xi(\theta), p(\theta)^{-1}p'(\theta)\rangle\, d\theta.$$

for $\xi \in \Omega\mathfrak{g}$. To check that this action preserves the product on $P_0\Omega G \times \mathrm{U}(1)$, we have to show that

$$\big(pf_1p^{-1},\ z_1\exp(ik\int_0^{2\pi} \beta_p(f_1(t)^{-1}f_1'(t))\, dt)\big) \cdot \big(pf_2p^{-1},\ z_2\exp(ik\int_0^{2\pi} \beta_p(f_2(t)^{-1}f_2'(t))\, dt)\big)$$

$$= \big(pf_1f_2p^{-1},\ z_1z_2\kappa(f_1,f_2)\exp(ik\int_0^{2\pi} \beta_p((f_1f_2)(t)^{-1}(f_1f_2)'(t))\, dt)\big).$$

It therefore suffices to establish the identity

$$\kappa(pf_1p^{-1}, pf_2p^{-1}) = \kappa(f_1, f_2)\exp\Big(ik\int_0^{2\pi} \big(\beta_p((f_1f_2)(t)^{-1}(f_1f_2)'(t)) - \beta_p(f_1(t)^{-1}f_1'(t)) - \beta_p(f_2(t)^{-1}f_2'(t))\big)\ dt\Big).$$

This is a straightforward computation that can safely be left to the reader.

Next we check that the normal subgroup N is preserved by the action of P_0G. For this we must show that if $(f, z) \in N$ then

$$\big(pfp^{-1},\ z\exp(ik\int_0^{2\pi} \beta_p(f^{-1}f')\, dt)\big) \in N.$$

Recall that N consists of pairs (γ, z) such that $\gamma \in \Omega^2 G$ and $z = \exp(-ik \int_{D_\gamma} \omega)$ where D_γ is a disk in ΩG with boundary γ. Therefore we need to show that

$$\exp\left(ik \int_{D_{p^{-1}\gamma p}} \omega\right) = \exp\left(ik \int_{D_\gamma} \omega\right) \exp\left(-ik \int_0^{2\pi} \beta_p(\gamma^{-1}\gamma')dt\right).$$

This follows immediately from the identity

$$\mathrm{Ad}(p)^* \omega = \omega - d\beta_p,$$

which is easily established by direct computation.

Finally, we have to check the formula for $d\alpha$. On passing to Lie algebras, diagram (10.13) gives rise to the following commutative diagram of Lie algebras:

$$\begin{tikzcd}
P_0\Omega\mathfrak{g} \oplus \mathbb{R} \arrow[r, "\overline{\mathrm{ev}}"] \arrow[d] & \Omega\mathfrak{g} \oplus \mathbb{R} \arrow[d] \\
P_0\Omega\mathfrak{g} \arrow[r, "\mathrm{ev}"] & \Omega\mathfrak{g}
\end{tikzcd}$$

where $\overline{\mathrm{ev}}$ is the homomorphism $(f, c) \mapsto (f(2\pi), c)$ for $f \in P_0\Omega\mathfrak{g}$ and $c \in \mathbb{R}$. To calculate $d\alpha(p)(\ell, c)$ we compute $\overline{\mathrm{ev}}(d\alpha(p)(\tilde{\ell}, c))$ where $\tilde{\ell}$ satisfies $\mathrm{ev}(\tilde{\ell}) = \ell$ (take, for example, $\tilde{\ell}(t) = t\ell/2\pi$). It is then straightforward to compute that

$$\overline{\mathrm{ev}}(d\alpha(p)(\tilde{\ell}, c)) = \big([p, \ell], 2k \int_0^{2\pi} \langle p(\theta), \ell'(\theta)\rangle \, d\theta\big).$$

$\square$

10.4 The 2-Group $\mathcal{P}_k G$ and $\mathrm{String}(n)$

Having completed our review of Lie 2-algebras and loop groups, we now study a Lie 2-group $\mathcal{P}_k G$ whose Lie 2-algebra $\mathcal{P}_k\mathfrak{g}$ is equivalent to $\mathfrak{g}_k$. We begin in Section 10.4.1 by giving a construction of $\mathcal{P}_k G$ in terms of the central extension $\widehat{\Omega_k G}$ of the loop group of G. This yields a description of $\mathcal{P}_k\mathfrak{g}$ which we use later to prove that this Lie 2-algebra is equivalent to $\mathfrak{g}_k$.

Section 10.4.2 gives another viewpoint on $\mathcal{P}_k G$, which goes a long way toward explaining the significance of this 2-group. For this, we study the topological group $|\mathcal{P}_k G|$ formed by taking the geometric realization of the nerve of $\mathcal{P}_k G$.

10.4.1 Constructing $\mathcal{P}_k G$

In Proposition 10.6 we saw that the action of the path group $P_0 G$ on the loop group ΩG by conjugation lifts to an action α of $P_0 G$ on the central extension $\widehat{\Omega_k G}$. This allows us to define a Fréchet Lie group $P_0 G \ltimes \widehat{\Omega_k G}$ in which multiplication is given by:

$$(p_1, \hat{\ell}_1) \cdot (p_2, \hat{\ell}_2) = \big(\, p_1 p_2,\ \hat{\ell}_1 \alpha(p_1)(\hat{\ell}_2)\,\big).$$

This, in turn, allows us to construct the 2-group $\mathcal{P}_k G$ which plays the starring role in this paper:

Proposition 10.7. *Suppose G is a simply-connected compact simple Lie group and $k \in \mathbb{Z}$. Then there is a Fréchet Lie 2-group $\mathcal{P}_k G$ for which:*

- *The Fréchet Lie group of objects* $\mathrm{Ob}(\mathcal{P}_k G)$ *is* $P_0 G$.
- *The Fréchet Lie group of morphisms* $\mathrm{Mor}(\mathcal{P}_k G)$ *is* $P_0 G \ltimes \widehat{\Omega_k G}$.
- *The source and target maps* $s, t\colon \mathrm{Mor}(\mathcal{P}_k G) \to \mathrm{Ob}(\mathcal{P}_k G)$ *are given by:*

$$\begin{aligned} s(p, \hat{\ell}) &= p \\ t(p, \hat{\ell}) &= \partial(\hat{\ell}) p \end{aligned}$$

  where $p \in P_0 G$, $\hat{\ell} \in \widehat{\Omega_k G}$, *and* $\partial\colon \widehat{\Omega_k G} \to P_0 G$ *is the composite:*

$$\widehat{\Omega_k G} \to \Omega G \hookrightarrow P_0 G\,.$$

- *The identity-assigning map* $i\colon \mathrm{Ob}(\mathcal{P}_k G) \to \mathrm{Mor}(\mathcal{P}_k G)$ *is given by:*

$$i(p) = (p, 1).$$

- *The composition map* $\circ\colon \mathrm{Mor}(\mathcal{P}_k G) \times_{\mathrm{Ob}(\mathcal{P}_k G)} \mathrm{Mor}(\mathcal{P}_k G) \to \mathrm{Mor}(\mathcal{P}_k G)$ *is given by:*

$$(p_1, \hat{\ell}_1) \circ (p_2, \hat{\ell}_2) = (p_2, \hat{\ell}_1 \hat{\ell}_2)$$

  whenever $(p_1, \hat{\ell}_1), (p_2, \hat{\ell}_2)$ *are composable morphisms in* $\mathcal{P}_k G$.

Proof. One can check directly that $s, t, i, \circ$ are Fréchet Lie group homomorphisms and that these operations make $\mathcal{P}_k G$ into a category. Alternatively, one can check that $(P_0 G, \widehat{\Omega_k G}, \alpha, \partial)$ is a crossed module in the category of Fréchet manifolds. This merely requires checking that

$$\partial(\alpha(p)(\hat{\ell})) = p\, \partial(\hat{\ell})\, p^{-1} \tag{10.14}$$

and

$$\alpha(\partial(\hat{\ell}_1))(\hat{\ell}_2) = \hat{\ell}_1 \hat{\ell}_2 \hat{\ell}_1^{-1}. \tag{10.15}$$

Then one can use the fact that crossed modules in the category of Fréchet manifolds are the same as Fréchet Lie 2-groups (see for example HDA6). □

We denote the Lie 2-algebra of $\mathcal{P}_k G$ by $\mathcal{P}_k \mathfrak{g}$. To prove this Lie 2-algebra is equivalent to $\mathfrak{g}_k$ in Section 10.5, we will use an explicit description of its corresponding L_∞-algebra:

Proposition 10.8. *The 2-term L_∞-algebra V corresponding to the Lie 2-algebra $\mathcal{P}_k \mathfrak{g}$ has:*

- $V_0 = P_0 \mathfrak{g}$ *and* $V_1 = \widehat{\Omega_k \mathfrak{g}} \cong \Omega \mathfrak{g} \oplus \mathbb{R}$,
- $d\colon V_1 \to V_0$ *equal to the composite*

$$\widehat{\Omega_k \mathfrak{g}} \to \Omega \mathfrak{g} \hookrightarrow P_0 \mathfrak{g}\,,$$

- $l_2\colon V_0 \times V_0 \to V_1$ *given by the bracket in* $P_0\mathfrak{g}$*:*

$$l_2(p_1, p_2) = [p_1, p_2],$$

and $l_2\colon V_0 \times V_1 \to V_1$ *given by the action* $d\alpha$ *of* $P_0\mathfrak{g}$ *on* $\widehat{\Omega_k\mathfrak{g}}$*, or explicitly:*

$$l_2(p, (\ell, c)) = \left([p, \ell],\ 2k \int_0^{2\pi} \langle p(\theta), \ell'(\theta)\rangle \, d\theta \right)$$

for all $p \in P_0\mathfrak{g}$*,* $\ell \in \Omega G$ *and* $c \in \mathbb{R}$*.*

- $l_3\colon V_0 \times V_0 \times V_0 \to V_1$ *equal to zero.*

Proof. This is a straightforward application of the correspondence described in Section 10.2.2. The formula for $l_2\colon V_0 \times V_1 \to V_1$ comes from Proposition 10.6, while $\ell_3 = 0$ because the Lie 2-algebra $\mathcal{P}_k\mathfrak{g}$ is strict. $\square$

10.4.2 The Topology of $|\mathcal{P}_k G|$

In this section we construct an exact sequence of Fréchet Lie 2-groups:

$$1 \to \mathcal{L}_k G \xrightarrow{\iota} \mathcal{P}_k G \xrightarrow{\pi} G \to 1\,,$$

where G is considered as a Fréchet Lie 2-group with only identity morphisms. Applying a certain procedure for turning topological 2-groups into topological groups, described below, we obtain this exact sequence of topological groups:

$$1 \to |\mathcal{L}_k G| \xrightarrow{|\iota|} |\mathcal{P}_k G| \xrightarrow{|\pi|} G \to 1\,.$$

Note that $|G| = G$. We then show that the topological group $|\mathcal{L}_k G|$ has the homotopy type of the Eilenberg–Mac Lane space $K(\mathbb{Z}, 2)$. Since $K(\mathbb{Z}, 2)$ is also the classifying space $B\mathrm{U}(1)$, the above exact sequence is a topological analogue of the exact sequence of Lie 2-algebras describing how $\mathfrak{g}_k$ is built from $\mathfrak{g}$ and $\mathfrak{u}(1)$:

$$0 \to \mathrm{b}\mathfrak{u}(1) \to \mathfrak{g}_k \to \mathfrak{g} \to 0\,,$$

where $\mathrm{b}\mathfrak{u}(1)$ is the Lie 2-algebra with a 0-dimensional space of objects and $\mathfrak{u}(1)$ as its space of morphisms.

The above exact sequence of topological groups exhibits $|\mathcal{P}_k G|$ as the total space of a principal $K(\mathbb{Z}, 2)$ bundle over G. Bundles of this sort are classified by their 'Dixmier–Douady class', which is an element of the integral third cohomology group of the base space. In the case at hand, this cohomology group is $H^3(G) \cong \mathbb{Z}$, generated by the element we called $[\nu/2\pi]$ in the Introduction. We shall show that the Dixmier–Douady class of the bundle $|\mathcal{P}_k G| \to G$ equals $k[\nu/2\pi]$. Using this, we show that for $k = \pm 1$, $|\mathcal{P}_k G|$ is a version of $\hat{G}$ — the topological group obtained from G by killing its third homotopy group.

We start by defining a map $\pi\colon \mathcal{P}_k G \to G$ as follows. We define π on objects $p \in \mathcal{P}_k G$ as follows:

$$\pi(p) = p(2\pi) \in G.$$

In other words, π applied to a based path in G gives the endpoint of this path. We define π on morphisms in the only way possible, sending any morphism $(p, \hat{\ell})\colon p \to \partial(\hat{\ell})p$ to the identity morphism on $\pi(p)$. It is easy to see that π is a **strict homomorphism** of Fréchet Lie 2-groups: in other words, a map that strictly preserves all the Fréchet Lie 2-group structure. Moreover, it is easy to see that π is onto both for objects and morphisms.

Next, we define the Fréchet Lie 2-group $\mathcal{L}_k G$ to be the **strict kernel** of π. In other words, the objects of $\mathcal{L}_k G$ are objects of $\mathcal{P}_k G$ that are mapped to 1 by π, and similarly for the morphisms of $\mathcal{L}_k G$, while the source, target, identity-assigning and composition maps for $\mathcal{L}_k G$ are just restrictions of those for $\mathcal{P}_k G$. So:

- the Fréchet Lie group of objects $\mathrm{Ob}(\mathcal{L}_k G)$ is ΩG,
- the Fréchet Lie group of morphisms $\mathrm{Mor}(\mathcal{L}_k G)$ is $\Omega G \ltimes \widehat{\Omega_k G}$,

where the semidirect product is formed using the action α restricted to ΩG. Moreover, the formulas for $s, t, i, \circ$ are just as in Proposition 10.7, but with loops replacing paths.

It is easy to see that the inclusion $\iota\colon \mathcal{L}_k G \to \mathcal{P}_k G$ is a strict homomorphism of Fréchet Lie 2-groups. We thus obtain:

Proposition 10.9. *The sequence of strict Fréchet 2-group homomorphisms*

$$1 \to \mathcal{L}_k G \xrightarrow{\iota} \mathcal{P}_k G \xrightarrow{\pi} G \to 1$$

is **strictly exact***, meaning that the image of each arrow is equal to the kernel of the next, both on objects and on morphisms.*

Any Fréchet Lie 2-group C is, among other things, a **topological category**: a category where the sets $\mathrm{Ob}(C)$ and $\mathrm{Mor}(C)$ are topological spaces and the source, target, identity-assigning and composition maps are continuous. Homotopy theorists have a standard procedure for taking the 'nerve' of a topological category and obtaining a simplicial space. They also know how to take the 'geometric realization' of any simplicial space, obtaining a topological space. We use $|C|$ to denote the geometric realization of the nerve of a topological category C. If C is in fact a topological 2-group — for example a Fréchet Lie 2-group — then $|C|$ naturally becomes a topological group.

For readers unfamiliar with these constructions, let us give a more hands-on description of how to build $|C|$. First for any $n \in \mathbb{N}$ we construct a space $|C|_n$. A point in $|C|_n$ consists of a string of n composable morphisms in C:

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} x_{n-1} \xrightarrow{f_n} x_n$$

together with a point in the standard n-simplex:

$$a \in \Delta_n = \{(a_0, \ldots, a_n) \in [0,1]\colon\ a_0 + \cdots + a_n = 1\}.$$

Since $|C|_n$ is a subset of $\mathrm{Mor}(C)^n \times \Delta_n$, we give it the subspace topology. There are well-known face maps $d_i\colon \Delta_n \to \Delta_{n+1}$ and degeneracies $s_i\colon \Delta_n \to \Delta_{n-1}$. We use these to build $|C|$ by gluing together all the spaces $|C|_n$ via the following identifications:

$$\Big(x_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} x_n, a\Big) \sim \Big(x_0 \xrightarrow{f_1} \cdots \xrightarrow{f_i} x_i \xrightarrow{1} x_i \xrightarrow{f_{i+1}} \cdots \xrightarrow{f_n} x_n, d_i(a)\Big)$$

for $0 \leq i \leq n$, and

$$\Big(x_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} x_n, a\Big) \sim \Big(x_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{i-2}} x_{i-1} \xrightarrow{f_i f_{i+1}} x_{i+1} \xrightarrow{f_{i+2}} \cdots \xrightarrow{f_n} x_n, s_i(a)\Big)$$

for $0 < i < n$, together with

$$\Big(x_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} x_n, a\Big) \sim \Big(x_1 \xrightarrow{f_2} x_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} x_n, s_0(a)\Big)$$

and

$$\Big(x_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} x_n, a\Big) \sim \Big(x_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-2}} x_{n-2} \xrightarrow{f_{n-1}} x_{n-1}, s_n(a)\Big)$$

This defines $|C|$ as a topological space, but when C is a topological 2-group the multiplication in C makes $|C|$ into a topological group. Moreover, if G is a topological group viewed as a topological 2-group with only identity morphisms, we have $|G| \cong G$.

Applying the functor $|\cdot|$ to the exact sequence in Proposition 10.9, we obtain this result, which implies Theorem 3:

Theorem 10.9. *The sequence of topological groups*

$$1 \to |\mathcal{L}_k G| \xrightarrow{|\iota|} |\mathcal{P}_k G| \xrightarrow{|\pi|} G \to 1$$

is exact, and $|\mathcal{L}_k G|$ has the homotopy type of $K(\mathbb{Z}, 2)$. Thus, $|\mathcal{P}_k G|$ is the total space of a $K(\mathbb{Z}, 2)$ bundle over G. The Dixmier–Douady class of this bundle is $k[\nu/2\pi] \in H^3(G)$. Moreover, $\mathcal{P}_k G$ is $\hat{G}$ when $k = \pm 1$.

Proof. It is easy to see directly that the functor $|\cdot|$ carries strictly exact sequences of topological 2-groups to exact sequences of topological groups. To show that $|\mathcal{L}_k G|$ is a $K(\mathbb{Z}, 2)$, we prove there is a strictly exact sequence of Fréchet Lie 2-groups

$$1 \to \mathrm{U}(1) \to \mathcal{E}\widehat{\Omega_k G} \to \mathcal{L}_k G \to 1\,. \tag{10.16}$$

Here $\mathrm{U}(1)$ is regarded as a Fréchet Lie 2-group with only identity morphisms, while $\mathcal{E}\widehat{\Omega_k G}$ is the Fréchet Lie 2-group with $\widehat{\Omega_k G}$ as its Fréchet Lie group of objects and precisely one morphism from any object to any other. In general:

Lemma 10.10. *For any Fréchet Lie group $\mathcal{G}$, there is a Fréchet Lie 2-group $\mathcal{E}\mathcal{G}$ with:*

- *$\mathcal{G}$ as its Fréchet Lie group of objects,*
- *$\mathcal{G} \ltimes \mathcal{G}$ as its Fréchet Lie group of morphisms, where the semidirect product is defined using the conjugation action of $\mathcal{G}$ on itself,*

and:

- *source and target maps given by* $s(g, g') = g$, $t(g, g') = gg'$,
- *identity-assigning map given by* $i(g) = (g, 1)$,
- *composition map given by* $(g_1, g_1') \circ (g_2, g_2') = (g_2, g_1' g_2')$ *whenever* (g_1, g_1'), (g_2, g_2') *are composable morphisms in* $\mathcal{EG}$.

Proof. It is straightforward to check that this gives a Fréchet Lie 2-group. Note that $\mathcal{EG}$ has $\mathcal{G}$ as objects and one morphism from any object to any other.
□

In fact, Segal [189] has already introduced $\mathcal{EG}$ under the name $\overline{\mathcal{G}}$, treating it as a topological category. He proved that $|\mathcal{EG}|$ is contractible. In fact, he exhibited $|\mathcal{EG}|$ as a model of $E\mathcal{G}$, the total space of the universal bundle over the classifying space $B\mathcal{G}$ of $\mathcal{G}$. Therefore, applying the functor $|\cdot|$ to the exact sequence (10.16), we obtain this short exact sequence of topological groups:

$$1 \to \mathrm{U}(1) \to E\widehat{\Omega_k G} \to |\mathcal{L}_k G| \to 1\,.$$

Since $E\widehat{\Omega_k G}$ is contractible, it follows that $|\mathcal{L}_k G| \cong E\widehat{\Omega G}/\mathrm{U}(1)$ has the homotopy type of $B\mathrm{U}(1) \simeq K(\mathbb{Z}, 2)$.

One can check that $|\pi|\colon |\mathcal{P}_k G| \to G$ is a locally trivial fiber bundle, so it defines a principal $K(\mathbb{Z}, 2)$ bundle over G. Like any such bundle, this is the pullback of the universal principal $K(\mathbb{Z}, 2)$ bundle $p\colon EK(\mathbb{Z}, 2) \to BK(\mathbb{Z}, 2)$ along some map $f\colon G \to BK(\mathbb{Z}, 2)$, giving a commutative diagram of spaces:

$$\begin{tikzcd}
|\mathcal{L}_k G| \arrow[r, "|\iota|"] \arrow[d, "\sim"'] & |\mathcal{P}_k G| \arrow[r, "|\pi|"] \arrow[d, "p^* f"] & G \arrow[d, "f"] \\
K(\mathbb{Z}, 2) \arrow[r, "i"] & EK(\mathbb{Z}, 2) \arrow[r, "p"] & BK(\mathbb{Z}, 2)
\end{tikzcd}$$

Indeed, such bundles are classified up to isomorphism by the homotopy class of f. Since $BK(\mathbb{Z}, 2) \simeq K(\mathbb{Z}, 3)$, this homotopy class is determined by the Dixmier–Douady class $f^*\kappa$, where κ is the generator of $H^3(K(\mathbb{Z}, 3)) \cong \mathbb{Z}$. The next order of business is to show that $f^*\kappa = k[\nu/2\pi]$.

For this, it suffices to show that f maps the generator of $\pi_3(G) \cong \mathbb{Z}$ to k times the generator of $\pi_3(K(\mathbb{Z}, 3)) \cong \mathbb{Z}$. Consider this bit of the long exact sequences of homotopy groups coming from the above diagram:

$$\begin{tikzcd}
\pi_3(G) \arrow[r, "\partial"] \arrow[d, "\pi_3(f)"'] & \pi_2(|\mathcal{L}_k G|) \arrow[d, "\cong"] \\
\pi_3(K(\mathbb{Z}, 3)) \arrow[r, "\partial'"] & \pi_2(K(\mathbb{Z}, 2))
\end{tikzcd}$$

Since the connecting homomorphism ∂' and the map from $\pi_2(|\mathcal{L}_k G|)$ to $\pi_2(K(\mathbb{Z},2))$ are isomorphisms, we can treat these as the identity by a suitable choice of generators. Thus, to show that $\pi_3(f)$ is multiplication by k it suffices to show this for the connecting homomorphism ∂.

To do so, consider this commuting diagram of Frechét Lie 2-groups:

$$\begin{tikzcd}
\Omega G \arrow[r, "\iota"] \arrow[d, "i"'] & P_0 G \arrow[r, "\pi"] \arrow[d, "i'"'] & G \arrow[d, "1"] \\
\mathcal{L}_k G \arrow[r, "\iota"] & \mathcal{P}_k G \arrow[r, "\pi"] & G
\end{tikzcd}$$

Here we regard the groups on top as 2-groups with only identity morphisms; the downwards-pointing arrows include these in the 2-groups on the bottom row. Applying the functor $|\cdot|$, we obtain a diagram where each row is a principal bundle:

$$\begin{tikzcd}
\Omega G \arrow[r, "|\iota|"] \arrow[d, "|i|"'] & P_0 G \arrow[r, "|\pi|"] \arrow[d, "|i'|"'] & G \arrow[d, "1"] \\
|\mathcal{L}_k G| \arrow[r, "|\iota|"] & |\mathcal{P}_k G| \arrow[r, "|\pi|"] & G
\end{tikzcd}$$

Taking long exact sequences of homotopy groups, this gives:

$$\begin{tikzcd}
\pi_3(G) \arrow[r, "1"] \arrow[d, "1"'] & \pi_2(\Omega G) \arrow[d, "\pi_2(|i|)"] \\
\pi_3(G) \arrow[r, "\partial"] & \pi_2(|\mathcal{L}_k G|)
\end{tikzcd}$$

Thus, to show that ∂ is multiplication by k it suffices to show this for $\pi_2(|i|)$.

For this, we consider yet another commuting diagram of Frechét Lie 2-groups:

$$\begin{tikzcd}
\mathrm{U}(1) \arrow[r] \arrow[d] & \widehat{\Omega_k G} \arrow[r] \arrow[d] & \Omega G \arrow[d, "i"'] \\
\mathrm{U}(1) \arrow[r] & \mathcal{E}\widehat{\Omega_k G} \arrow[r] & \mathcal{L}_k G
\end{tikzcd}$$

Applying $|\cdot|$, we obtain a diagram where each row is a principal $\mathrm{U}(1)$ bundle:

$$\begin{tikzcd}
\mathrm{U}(1) \arrow[r] \arrow[d] & \widehat{\Omega_k G} \arrow[r] \arrow[d] & \Omega G \arrow[d, "|i|"] \\
\mathrm{U}(1) \arrow[r] & |\mathcal{E}\widehat{\Omega_k G}| \arrow[r] & |\mathcal{L}_k G| \simeq K(\mathbb{Z},2)
\end{tikzcd}$$

Recall that the bottom row is the universal principal U(1) bundle. The arrow $|i|$ is the classifying map for the U(1) bundle $\widehat{\Omega_k G} \to \Omega G$. By Theorem 10.8, the Chern class of this bundle is k times the generator of $H^2(\Omega G)$, so $\pi_2(|i|)$ must map the generator of $\pi_2(\Omega G)$ to k times the generator of $\pi_2(K(\mathbb{Z},2))$.

Finally, let us show that $|\mathcal{P}_k G|$ is $\hat{G}$ when $k = \pm 1$. For this, it suffices to show that when $k = \pm 1$, the map $|\pi| \colon |\mathcal{P}_k G| \to G$ induces isomorphisms on all homotopy groups except the third, and that $\pi_3(|\mathcal{P}_k G|) = 0$. For this we examine the long exact sequence:

$$\cdots \longrightarrow \pi_n(|\mathcal{L}_k G|) \longrightarrow \pi_n(|\mathcal{P}_k G|) \longrightarrow \pi_n(G) \xrightarrow{\partial} \pi_{n-1}(|\mathcal{L}_k G|) \longrightarrow \cdots .$$

Since $|\mathcal{L}_k G| \simeq K(\mathbb{Z},2)$, its homotopy groups vanish except for $\pi_2(|\mathcal{L}_k G|) \cong \mathbb{Z}$, so $|\pi|$ induces an isomorphism on π_n except possibly for $n = 2, 3$. In this portion of the long exact sequence we have

$$0 \longrightarrow \pi_3(|\mathcal{P}_k G|) \longrightarrow \mathbb{Z} \xrightarrow{k} \mathbb{Z} \longrightarrow \pi_2(|\mathcal{P}_k G|) \longrightarrow 0$$

so $\pi_3(|\mathcal{P}_k G|) \cong 0$ unless $k = 0$, and $\pi_2(|\mathcal{P}_k G|) \cong \mathbb{Z}/k\mathbb{Z}$, so $\pi_2(|\mathcal{P}_k G|) \cong \pi_2(G) \cong 0$ when $k = \pm 1$. $\square$

10.5 The Equivalence between $\mathcal{P}_k\mathfrak{g}$ and $\mathfrak{g}_k$

In this section we prove our main result, which implies Theorem 2:

Theorem 10.11. *There is a strictly exact sequence of Lie 2-algebra homomorphisms*

$$0 \to \mathcal{E}\Omega\mathfrak{g} \xrightarrow{\lambda} \mathcal{P}_k\mathfrak{g} \xrightarrow{\phi} \mathfrak{g}_k \to 0$$

where $\mathcal{E}\Omega\mathfrak{g}$ is equivalent to the trivial Lie 2-algebra and ϕ is an equivalence of Lie 2-algebras.

Recall that by 'strictly exact' we mean that both on the vector spaces of objects and the vector spaces of morphisms, the image of each map is the kernel of the next.

We prove this result in a series of lemmas. We begin by describing $\mathcal{E}\Omega\mathfrak{g}$ and showing that it is equivalent to the trivial Lie 2-algebra. Recall that in Lemma 10.10 we constructed for any Fréchet Lie group $\mathcal{G}$ a Fréchet Lie 2-group $\mathcal{E}\mathcal{G}$ with $\mathcal{G}$ as its group of objects and precisely one morphism from any object to any other. We saw that the space $|\mathcal{E}\mathcal{G}|$ is contractible; this is a topological reflection of the fact that $\mathcal{E}\mathcal{G}$ is equivalent to the trivial Lie 2-group. Now we need the Lie algebra analogue of this construction:

Lemma 10.12. *Given a Lie algebra L, there is a 2-term L_∞-algebra V for which:*

- $V_0 = L$ *and* $V_1 = L$,
- $d \colon V_1 \to V_0$ *is the identity,*
- $l_2 \colon V_0 \times V_0 \to V_1$ *and* $l_2 \colon V_0 \times V_1 \to V_1$ *are given by the bracket in* L,

- $l_3\colon V_0 \times V_0 \times V_0 \to V_1$ *is equal to zero.*

We call the corresponding strict Lie 2-algebra $\mathcal{E}L$.

Proof. Straightforward. □

Lemma 10.13. *For any Lie algebra L, the Lie 2-algebra $\mathcal{E}L$ is equivalent to the trivial Lie 2-algebra. That is, $\mathcal{E}L \simeq 0$.*

Proof. There is a unique homomorphism $\beta\colon \mathcal{E}L \to 0$ and a unique homomorphism $\gamma\colon 0 \to \mathcal{E}L$. Clearly $\beta \circ \gamma$ equals the identity. The composite $\gamma \circ \beta$ has:

$$\begin{array}{lcl} (\gamma\circ\beta)_0\colon & x & \mapsto 0 \\ (\gamma\circ\beta)_1\colon & x & \mapsto 0 \\ (\gamma\circ\beta)_2\colon & (x_1,x_2) & \mapsto 0\,, \end{array}$$

while the identity homomorphism from $\mathcal{E}L$ to itself has:

$$\begin{array}{lcl} \mathrm{id}_0\colon & x & \mapsto x \\ \mathrm{id}_1\colon & x & \mapsto x \\ \mathrm{id}_2\colon & (x_1,x_2) & \mapsto 0\,. \end{array}$$

There is a 2-isomorphism

$$\tau\colon \gamma\circ\beta \stackrel{\sim}{\Longrightarrow} \mathrm{id}$$

given by

$$\tau(x) = x\,,$$

where the x on the left is in V_0 and that on the right in V_1, but of course $V_0 = V_1$ here. □

We continue by defining the Lie 2-algebra homomorphism $\mathcal{P}_k\mathfrak{g} \xrightarrow{\phi} \mathfrak{g}_k$.

Lemma 10.14. *There exists a Lie 2-algebra homomorphism*

$$\phi\colon \mathcal{P}_k\mathfrak{g} \to \mathfrak{g}_k$$

which we describe in terms of its corresponding L_∞-homomorphism:

$$\begin{aligned} \phi_0(p) &= p(2\pi) \\ \phi_1(\ell,c) &= c \\ \phi_2(p_1,p_2) &= k\int_0^{2\pi} \left(\langle p_1|p_2'\rangle - \langle p_1'|p_2\rangle\right)\,d\theta \end{aligned}$$

where $p, p_1, p_2 \in P_0\mathfrak{g}$ and $(\ell,c) \in \Omega\mathfrak{g}\oplus\mathbb{R} \cong \widehat{\Omega_k\mathfrak{g}}$.

Before beginning, note that the quantity

$$\int_0^{2\pi} \left(\langle p_1|p_2'\rangle - \langle p_1'|p_2\rangle\right)\, d\theta = 2\int_0^{2\pi} \langle p_1|p_2'\rangle\ d\theta \ - \ \langle p_1(2\pi)|p_2(2\pi)\rangle$$

is skew-symmetric, but not in general equal to

$$2\int_0^{2\pi} \langle p_1|p_2'\rangle\ d\theta$$

due to the boundary term. However, these quantities are equal when either p_1 or p_2 is a loop.

Proof. We must check that ϕ satisfies the conditions of Definition 10.6. First we show that ϕ is a chain map. That is, we show that ϕ_0 and ϕ_1 preserve the differentials:

$$\begin{tikzcd}
\widehat{\Omega_k\mathfrak{g}} \arrow[r, "d"] \arrow[d, "\phi_1"'] & P_0\mathfrak{g} \arrow[d, "\phi_0"] \\
\mathbb{R} \arrow[r, "d'"] & \mathfrak{g}
\end{tikzcd}$$

where d is the composite given in Proposition 10.8, and $d' = 0$ since $\mathfrak{g}_k$ is skeletal. This square commutes since ϕ_0 is also zero.

We continue by verifying conditions (10.3) - (10.5) of Definition 10.6. The bracket on objects is preserved on the nose, which implies that the right-hand side of (10.3) is zero. This is consistent with the fact that the differential in the L_∞-algebra for $\mathfrak{g}_k$ is zero, which implies that the left-hand side of (10.3) is also zero.

The right-hand side of (10.4) is given by:

$$\begin{aligned}
\phi_1(l_2(p,(\ell,c)) - l_2(\phi_0(p),\phi_1(\ell,c)) &= \phi_1\Big([p,\ell], 2k\int \langle p|\ell'\rangle\ d\theta\Big) \ - \underbrace{l_2(p(2\pi),c)}_{=0} \\
&= 2k\int \langle p|\ell'\rangle\ d\theta.
\end{aligned}$$

This matches the left-hand side of (10.4), namely:

$$\begin{aligned}
\phi_2(p, d(\ell,c)) &= \phi_2(p,\ell) \\
&= k\int (\langle p|\ell'\rangle - \langle p'|\ell\rangle)\, d\theta \\
&= 2k\int \langle p|\ell'\rangle\ d\theta
\end{aligned}$$

Note that no boundary term appears here since one of the arguments is a loop.

Finally, we check condition (10.5). Four terms in this equation vanish because $l_3 = 0$ in $\mathcal{P}_k\mathfrak{g}$ and $l_2 = 0$ in $\mathfrak{g}_k$. We are left needing to show

$$l_3(\phi_0(p_1),\phi_0(p_2),\phi_0(p_3)) = \phi_2(p_1, l_2(p_2,p_3)) + \phi_2(p_2, l_2(p_3,p_1)) + \phi_2(p_3, l_2(p_1,p_2))\,.$$

The left-hand side here equals $k\langle p_1(2\pi), [p_2(2\pi), p_3(2\pi)]\rangle$. The right-hand side equals:

$$\begin{aligned} & \phi_2(p_1, l_2(p_2, p_3)) \quad + \quad \text{cyclic permutations} \\ = \; & k\int \left(\langle p_1 | [p_2, p_3]' \rangle - \langle p_1' | [p_2, p_3] \rangle\right) \, d\theta \quad + \text{ cyclic perms.} \\ = \; & k\int \left(\langle p_1 | [p_2', p_3] \rangle + \langle p_1 | [p_2, p_3'] \rangle - \langle p_1' | [p_2, p_3] \rangle\right) \, d\theta \quad + \quad \text{cyclic perms.} \end{aligned}$$

Using the antisymmetry of $\langle \cdot, [\cdot, \cdot] \rangle$, this becomes:

$$k\int \left(\langle p_2' | [p_3, p_1] \rangle + \langle p_3' | [p_1, p_2] \rangle - \langle p_1' | [p_2, p_3] \rangle\right) \, d\theta \quad + \quad \text{cyclic perms.}$$

The last two terms cancel when we add all their cyclic permutations, so we are left with all three cyclic permutations of the first term:

$$k\int \left(\langle p_1' | [p_2, p_3] \rangle + \langle p_2' | [p_3, p_1] \rangle + \langle p_3' | [p_1, p_2] \rangle\right) \, d\theta \, .$$

If we apply integration by parts to the first term, we get:

$$\begin{aligned} k\int \Big(- \langle p_1 | [p_2', p_3] \rangle - \langle p_1 | [p_2, p_3'] \rangle + \langle p_2' | [p_3, p_1] \rangle + \langle p_3' | [p_1, p_2] \rangle \Big) \; d\theta \, + \\ k \, \langle p_1(2\pi) | [p_2(2\pi), p_3(2\pi)] \rangle \; . \end{aligned}$$

By the antisymmetry of $\langle \cdot, [\cdot, \cdot] \rangle$, the four terms in the integral cancel, leaving just $k\langle p_1(2\pi), [p_2(2\pi), p_3(2\pi)]\rangle$, as desired. □

Next we show that the strict kernel of $\phi\colon \mathcal{P}_k\mathfrak{g} \to \mathfrak{g}_k$ is $\mathcal{E}\Omega\mathfrak{g}$:

Lemma 10.15. *There is a Lie 2-algebra homomorphism*

$$\lambda\colon \mathcal{E}\Omega\mathfrak{g} \to \mathcal{P}_k\mathfrak{g},$$

that is one-to-one both on objects and on morphisms, and whose range is precisely the kernel of $\phi\colon \mathcal{P}_k\mathfrak{g} \to \mathfrak{g}_k$, both on objects and on morphisms.

Proof. Glancing at the formula for ϕ in Lemma 10.14, we see that the kernel of ϕ_0 and the kernel of ϕ_1 are both $\Omega\mathfrak{g}$. We see from Lemma 10.12 that these are precisely the spaces V_0 and V_1 in the 2-term L_∞-algebra V corresponding to $\mathcal{E}\Omega\mathfrak{g}$. The differential $d\colon \ker(\phi_1) \to \ker(\phi_0)$ inherited from $\mathcal{E}\Omega\mathfrak{g}$ also matches that in V: it is the identity map on $\Omega\mathfrak{g}$.

Thus, we obtain an inclusion of 2-vector spaces $\lambda\colon \mathcal{E}\Omega\mathfrak{g} \to \mathcal{P}_k\mathfrak{g}$. This uniquely extends to a Lie 2-algebra homomorphism, which we describe in terms of its corresponding L_∞-homomorphism:

$$\lambda_0(\ell) = \ell$$

$$\begin{aligned}\lambda_1(\ell) &= (\ell, 0)\\ \lambda_2(\ell_1,\ell_2) &= \Big(0, -2k \int_0^{2\pi} \langle \ell_1 | \ell_2' \rangle \, d\theta\Big)\end{aligned}$$

where $\ell, \ell_1, \ell_2 \in \Omega\mathfrak{g}$, and the zero in the last line denotes the zero loop.

To prove this, we must show that the conditions of Definition 10.6 are satisfied. We first check that λ is a chain map, i.e., this square commutes:

$$\begin{tikzcd} \Omega\mathfrak{g} \arrow[r, "d"] \arrow[d, "\lambda_1"'] & \Omega\mathfrak{g} \arrow[d, "\lambda_0"] \\ \widehat{\Omega_k\mathfrak{g}} \arrow[r, "d'"] & P_0\mathfrak{g} \end{tikzcd}$$

where d is the identity and d' is the composite given in Proposition 10.8. To see this, note that $d'(\lambda_1(\ell)) = d'(\ell, 0) = \ell$ and $\lambda_0(d(\ell)) = \lambda_0(\ell) = \ell$.

We continue by verifying conditions (10.3) - (10.5) of Definition 10.6. The bracket on the space V_0 is strictly preserved by λ_0, which implies that the right-hand side of (10.3) is zero. It remains to show that the left-hand side, $d'(\lambda_2(\ell_1, \ell_2))$, is also zero. Indeed, we have:

$$d'(\lambda_2(\ell_1,\ell_2)) = d'\left(0, -2k \int \langle \ell_1 | \ell_2' \rangle \, d\theta\right) = 0\,.$$

Next we check property (10.4). On the right-hand side, we have:

$$\begin{aligned}\lambda_1(l_2(\ell_1,\ell_2)) - l_2(\lambda_0(\ell_1), \lambda_1(\ell_2)) &= ([\ell_1,\ell_2], 0) - \Big([\ell_1,\ell_2], 2k \int \langle \ell_1 | \ell_2' \rangle \, d\theta\Big)\\ &= \Big(0, -2k \int \langle \ell_1 | \ell_2' \rangle \, d\theta\Big)\,.\end{aligned}$$

On the left-hand side, we have:

$$\lambda_2(\ell_1, d(\ell_2)) = \lambda_2(\ell_1,\ell_2) = \Big(0, -2k \int \langle \ell_1 | \ell_2' \rangle \, d\theta\Big)$$

Note that this also shows that given the chain map defined by λ_0 and λ_1, the function λ_2 that extends this chain map to an L_∞-homomorphisms is unquely fixed by condition (10.4).

Finally, we show that λ_2 satisfies condition (10.5). The two terms involving l_3 vanish since λ is a map between two strict Lie 2-algebras. The three terms of the form $l_2(\lambda_0(\cdot), \lambda_2(\cdot,\cdot))$ vanish because the image of λ_2 lies in the center of $\widehat{\Omega_k\mathfrak{g}}$. It thus remains to show that

$$\lambda_2(\ell_1, l_2(\ell_2,\ell_3)) + \lambda_2(\ell_2, l_2(\ell_3,\ell_1)) + \lambda_2(\ell_3, l_2(\ell_1,\ell_2)) = 0.$$

This is just the cocycle property of the Kac–Moody cocycle, Equation (10.10).
□

Next we check the exactness of the sequence

$$0 \to \mathcal{E}\Omega\mathfrak{g} \xrightarrow{\lambda} \mathcal{P}_k\mathfrak{g} \xrightarrow{\phi} \mathfrak{g}_k \to 0$$

at the middle point. Before doing so, we recall the formulas for the L_∞-homomorphisms corresponding to λ and ϕ. The L_∞-homomorphism corresponding to $\lambda\colon \mathcal{E}\Omega\mathfrak{g} \to \mathcal{P}_k\mathfrak{g}$ is given by

$$\begin{aligned} \lambda_0(\ell) &= \ell \\ \lambda_1(\ell) &= (\ell, 0) \\ \lambda_2(\ell_1, \ell_2) &= \Big(0,\ -2k \int_0^{2\pi} \langle \ell_1 | \ell_2' \rangle \, d\theta \Big) \end{aligned}$$

where $\ell, \ell_1, \ell_2 \in \Omega\mathfrak{g}$, and that corresponding to $\phi\colon \mathcal{P}_k\mathfrak{g} \to \mathfrak{g}_k$ is given by:

$$\begin{aligned} \phi_0(p) &= p(2\pi) \\ \phi_1(\ell, c) &= c \\ \phi_2(p_1, p_2) &= k \int_0^{2\pi} \big(\langle p_1 | p_2' \rangle - \langle p_1' | p_2 \rangle \big) \, d\theta \end{aligned}$$

where $p, p_1, p_2 \in P_0\mathfrak{g}$, $\ell \in \Omega\mathfrak{g}$, and $c \in \mathbb{R}$.

Lemma 10.16. *The composite*

$$\mathcal{E}\Omega\mathfrak{g} \xrightarrow{\lambda} \mathcal{P}_k\mathfrak{g} \xrightarrow{\phi} \mathfrak{g}_k$$

is the zero homomorphism, and the kernel of ϕ is precisely the image of λ, both on objects and on morphisms.

Proof. The composites $(\phi\circ\lambda)_0$ and $(\phi\circ\lambda)_1$ clearly vanish. Moreover $(\phi\circ\lambda)_2$ vanishes since:

$$\begin{aligned} (\phi \circ \lambda)_2(\ell_1, \ell_2) &= \phi_2(\lambda_0(\ell_1), \lambda_0(\ell_2)) + \phi_1(\lambda_2(\ell_1, \ell_2)) \qquad \text{by (10.6)} \\ &= \phi_2(\ell_1, \ell_2) + \phi_1\Big(0, -2k \int \langle \ell_1 | \ell_2' \rangle \, d\theta\Big) \\ &= k \int (\langle \ell_1 | \ell_2' \rangle - \langle \ell_1' | \ell_2 \rangle) \, d\theta - 2k \int \langle \ell_1 | \ell_2' \rangle \, d\theta \\ &= 0 \end{aligned}$$

with the help of integration by parts. The image of λ is precisely the kernel of ϕ by construction. ☐

Note that ϕ is obviously onto, both for objects and morphisms, so we have an exact sequence

$$0 \to \mathcal{E}\Omega\mathfrak{g} \xrightarrow{\lambda} \mathcal{P}_k\mathfrak{g} \xrightarrow{\phi} \mathfrak{g}_k \to 0\,.$$

Next we construct a family of splittings $\psi\colon \mathfrak{g}_k \to \mathcal{P}_k\mathfrak{g}$ for this exact sequence:

Lemma 10.17. *Suppose*

$$f\colon [0,2\pi] \to \mathbb{R}$$

is a smooth function with $f(0) = 0$ and $f(2\pi) = 1$. Then there is a Lie 2-algebra homomorphism

$$\psi\colon \mathfrak{g}_k \to \mathcal{P}_k\mathfrak{g}$$

whose corresponding L_∞-homomorphism is given by:

$$\begin{aligned} \psi_0(x) &= xf \\ \psi_1(c) &= (0,c) \\ \psi_2(x_1,x_2) &= ([x_1,x_2](f-f^2),0) \end{aligned}$$

where $x, x_1, x_2 \in \mathfrak{g}$ and $c \in \mathbb{R}$.

Proof. We show that ψ satisfies the conditions of Definition 10.6. We begin by showing that ψ is a chain map, meaning that the following square commutes:

$$\begin{tikzcd} \mathbb{R} \arrow[r, "d"] \arrow[d, "\psi_1"'] & \mathfrak{g} \arrow[d, "\psi_0"] \\ \widehat{\Omega_k\mathfrak{g}} \arrow[r, "d'"] & P_0\mathfrak{g} \end{tikzcd}$$

where $d = 0$ since $\mathfrak{g}_k$ is skeletal and d' is the composite given in Proposition 10.8. This square commutes because $\psi_0(d(c)) = \psi_0(0) = 0$ and $d'(\psi_1(c)) = d'(0,c) = 0$.

We continue by verifying conditions (10.3) - (10.5) of Definition 10.6. The right-hand side of (10.3) equals:

$$\psi_0(l_2(x_1,x_2)) - l_2(\psi_0(x_1),\psi_0(x_2)) = [x_1,x_2](f-f^2)\,.$$

This equals the left-hand side $d'(\psi_2(x_1,x_2))$ by construction.

The right-hand side of (10.4) equals:

$$\psi_1(l_2(x,c)) - l_2(\psi_0(x),\psi_1(c)) \;=\; \psi_1(0) - l_2(xf,(0,c)) \;=\; 0$$

since both terms vanish separately. Since the left-hand side is $\psi_2(x,dc) = \psi_2(x,0) = 0$, this shows that ψ satisfies condition (10.4).

Finally we verify condition (10.5). The term $l_3(\psi_0(\cdot),\psi_0(\cdot),\psi_0(\cdot))$ vanishes because $\mathcal{P}_k\mathfrak{g}$ is strict. The sum of three other terms vanishes thanks to the Jacobi identity in $\mathfrak{g}$:

$$\begin{aligned} &\psi_2(x_1,l_2(x_2,x_3)) + \psi_2(x_2,l_2(x_3,x_1)) + \psi_2(x_3,l_2(x_1,x_2)) \\ &= \Big(([x_1,[x_2,x_3]] + [x_2,[x_3,x_1]] + [x_3,[x_1,x_2]])\,(f-f^2),\,0\Big) \\ &= (0,0)\,. \end{aligned}$$

Thus, it remains to show that:

$$-\psi_1(l_3(x_1,x_2,x_3)) =$$

$$l_2(\psi_0(x_1),\psi_2(x_2,x_3)) + l_2(\psi_0(x_2),\psi_2(x_3,x_1)) + l_2(\psi_0(x_3),\psi_2(x_1,x_2))\,.$$

This goes as follows:

$$\begin{aligned}
& l_2(\psi_0(x_1),\psi_2(x_2,x_3)) + l_2(\psi_0(x_2),\psi_2(x_3,x_1)) + l_2(\psi_0(x_3),\psi_2(x_1,x_2)) \\
&= \Big(0, 3\cdot 2k \int_0^{2\pi} \langle x_1|[x_2,x_3]\rangle \; f(f-f^2)'\, d\theta \;\Big) \\
&= (0, -k\,\langle x_1|[x_2,x_3]\rangle) \qquad \text{by the calculation below} \\
&= -\psi_1(l_3(x_1,x_2,x_3))\,.
\end{aligned}$$

The value of the integral here is *universal*, independent of the choice of f:

$$\begin{aligned}
\int_0^{2\pi} f(f-f^2)'\, d\theta &= \int_0^{2\pi} \big(f(\theta)f'(\theta) - 2f^2(\theta)f'(\theta)\big)\; d\theta \\
&= \frac{1}{2} - \frac{2}{3} = -\frac{1}{6}\,.
\end{aligned}$$

□

The final step in proving Theorem 10.11 is to show that $\phi\circ\psi$ is the identity on $\mathfrak{g}_k$, while $\psi\circ\phi$ is isomorphic to the identity on $\mathcal{P}_k\mathfrak{g}$. For convenience, we recall the definitions first: $\phi\colon \mathcal{P}_k\mathfrak{g}\to\mathfrak{g}_k$ is given by:

$$\begin{aligned}
\phi_0(p) &= p(2\pi) \\
\phi_1(\ell,c) &= c \\
\phi_2(p_1,p_2) &= k\int_0^{2\pi} \big(\langle p_1|p_2'\rangle - \langle p_1'|p_2\rangle\big)\; d\theta
\end{aligned}$$

where $p,p_1,p_2\in P_0\mathfrak{g}$, $\ell\in\widehat{\Omega_k\mathfrak{g}}$, and $c\in\mathbb{R}$, while $\psi\colon \mathfrak{g}_k\to\mathcal{P}_k\mathfrak{g}$ is given by:

$$\begin{aligned}
\psi_0(x) &= xf \\
\psi_1(c) &= (0,c) \\
\psi_2(x_1,x_2) &= ([x_1,x_2](f-f^2),0)
\end{aligned}$$

where $x,x_1,x_2\in\mathfrak{g}$, $c\in\mathbb{R}$, and $f\colon[0,2\pi]\to\mathbb{R}$ satisfies the conditions of Lemma 10.17.

Lemma 10.18. *With the above definitions we have:*

- *$\phi\circ\psi$ is the identity Lie 2-algebra homomorphism on $\mathfrak{g}_k$;*
- *$\psi\circ\phi$ is isomorphic, as a Lie 2-algebra homomorphism, to the identity on $\mathcal{P}_k\mathfrak{g}$.*

Proof. We begin by demonstrating that $\phi \circ \psi$ is the identity on $\mathfrak{g}_k$. First,

$$(\phi \circ \psi)_0(x) = \phi_0(\psi_0(x)) = \phi_0(xf) = xf(2\pi) = x,$$

since $f(2\pi) = 1$ by the definition of f in Lemma 10.17. Second,

$$(\phi \circ \psi)_1(c) = \phi_1(\psi_1(c)) = \phi_1((0, c)) = c$$

Finally,

$$\begin{aligned}
(\phi \circ \psi)_2(x_1, x_2) &= \phi_2(\psi_0(x_1), \psi_0(x_2)) + \phi_1(\psi_2(x_1, x_2)) \qquad \text{by (10.6)} \\
&= \phi_2(x_1 f, x_2 f) \; + \; \phi_1([x_1, x_2](f - f^2), 0) \\
&= k \int (\langle x_1 f | x_2 f' \rangle - \langle x_1 f' | x_2 f \rangle) \, d\theta \; + \; 0 \\
&= k \, \langle x_1 | x_2 \rangle \int (f f' - f' f) \, d\theta \\
&= 0 \, .
\end{aligned}$$

Next we consider the composite

$$\psi \circ \phi \colon \mathcal{P}_k \mathfrak{g} \to \mathcal{P}_k \mathfrak{g} \, .$$

The corresponding L_∞-algebra homomorphism is given by:

$$\begin{aligned}
(\psi \circ \phi)_0(p) &= p(2\pi) f \\
(\psi \circ \phi)_1(\ell, c) &= (0, c) \\
(\psi \circ \phi)_2(p_1, p_2) &= \left([p_1(2\pi), p_2(2\pi)](f - f^2), \; k \int \left(\langle p_1 | p_2' \rangle - \langle p_1' | p_2 \rangle \right) \, d\theta \right)
\end{aligned}$$

where again we use equation (10.6) to obtain the formula for $(\psi \circ \phi)_2$.

For this to be isomorphic to the identity there must exist a Lie 2-algebra 2-isomorphism

$$\tau \colon \psi \circ \phi \Longrightarrow \mathrm{id}$$

where id is the identity on $\mathcal{P}_k \mathfrak{g}$. We define this in terms of its corresponding L_∞-2-homomorphism by setting:

$$\tau(p) = (p - p(2\pi) f, 0) \, .$$

Thus, τ turns a path p into the loop $p - p(2\pi) f$.

We must show that τ is a chain homotopy satisfying condition (10.7) of Definition 10.7. We begin by showing that τ is a chain homotopy. We have

$$\begin{aligned}
d(\tau(p)) \; = \; d(p - p(2\pi) f, 0) &= p - p(2\pi) f \\
&= \mathrm{id}_0(p) - (\psi \circ \phi)_0(p)
\end{aligned}$$

and

$$\begin{aligned}
\tau(d(\ell, c)) \; = \; \tau(\ell) &= (\ell, 0) \\
&= \mathrm{id}_1(\ell, c) - (\psi \circ \phi)_1(\ell, c)
\end{aligned}$$

so τ is indeed a chain homotopy.

We conclude by showing that τ satisfies condition (10.7):

$$(\psi \circ \phi)_2(p_1, p_2) = l_2((\psi \circ \phi)_0(p_1), \tau(p_2)) + l_2(\tau(p_1), p_2) - \tau(l_2(p_1, p_2))$$

In order to verify this equation, we write out the right-hand side more explicitly by inserting the formulas for $(\psi \circ \phi)_2$ and for τ, obtaining:

$$l_2\big(p_1(2\pi)f,\, (p_2 - p_2(2\pi)f, 0)\big) + l_2\big((p_1 - p_1(2\pi)f, 0),\, p_2\big) - ([p_1, p_2] - [p_1(2\pi), p_2(2\pi)]f,\, 0)$$

This is an ordered pair consisting of a loop in $\mathfrak{g}$ and a real number. By collecting summands, the loop itself turns out to be:

$$[p_1(2\pi), p_2(2\pi)](f - f^2)\,.$$

Similarly, after some integration by parts the real number is found to be:

$$k \int_0^{2\pi} \left(\langle p_1 | p_2' \rangle - \langle p_1' | p_2 \rangle\right)\, d\theta\,.$$

Comparing these results with the value of $(\psi \circ \phi)_2(p_1, p_2)$ given above, one sees that τ indeed satisfies (10.7). $\square$

10.6 Conclusions

We have seen that the Lie 2-algebra $\mathfrak{g}_k$ is equivalent to an infinite-dimensional Lie 2-algebra $\mathcal{P}_k\mathfrak{g}$, and that when k is an integer, $\mathcal{P}_k\mathfrak{g}$ comes from an infinite-dimensional Lie 2-group $\mathcal{P}_kG$. Just as the Lie 2-algebra $\mathfrak{g}_k$ is built from the simple Lie algebra $\mathfrak{g}$ and a shifted version of $\mathfrak{u}(1)$:

$$0 \longrightarrow \mathrm{b}\mathfrak{u}(1) \longrightarrow \mathfrak{g}_k \longrightarrow \mathfrak{g} \longrightarrow 0\,,$$

the Lie 2-group $\mathcal{P}_kG$ is built from G and another Lie 2-group:

$$1 \longrightarrow \mathcal{L}_kG \longrightarrow \mathcal{P}_kG \longrightarrow G \longrightarrow 1$$

whose geometric realization is a shifted version of $\mathrm{U}(1)$:

$$1 \longrightarrow B\mathrm{U}(1) \longrightarrow |\mathcal{P}_kG| \longrightarrow G \longrightarrow 1\,.$$

None of these exact sequences split; in every case an interesting cocycle plays a role in defining the middle term. In the first case, the Jacobiator of $\mathfrak{g}_k$ is $k\nu\colon \Lambda^3\mathfrak{g} \to \mathbb{R}$. In the second case, composition of morphisms is defined using multiplication in the level-k Kac–Moody central extension of ΩG, which relies on the Kac–Moody cocycle $k\omega\colon \Lambda^2\Omega\mathfrak{g} \to \mathbb{R}$. In the third case, $|\mathcal{P}_kG|$ is the total space of a twisted $B\mathrm{U}(1)$-bundle over G whose Dixmier–Douady class is $k[\nu/2\pi] \in H^3(G)$. Of course, all these cocycles are different manifestations of the fact that every simply-connected compact simple Lie algebra has $H^3(G) = \mathbb{Z}$.

We conclude with some remarks of a more speculative nature. There is a theory of '2-bundles' in which a Lie 2-group plays the role of structure group [31, 36]. Connections on 2-bundles describe parallel transport of 1-dimensional extended objects, e.g. strings. Given the importance of the Kac–Moody extensions of loop groups in string theory, it is natural to guess that connections on 2-bundles with structure group $\mathcal{P}_k G$ will play a role in this theory.

The case when $G = \mathrm{Spin}(n)$ and $k = 1$ is particularly interesting, since then $|\mathcal{P}_k G| = \mathrm{String}(n)$. In this case we suspect that 2-bundles on a spin manifold M with structure 2-group $\mathcal{P}_k G$ can be thought as substitutes for principal $\mathrm{String}(n)$-bundles on M. It is interesting to think about 'string structures' [87] on M from this perspective: given a principal G-bundle P on M (thought of as a 2-bundle with only identity morphisms) one can consider the obstruction problem of trying to lift the structure 2-group from G to $\mathcal{P}_k G$. There should be a single topological obstruction in $H^4(M;\mathbb{Z})$ to finding a lift, namely the characteristic class $p_1/2$. When this characteristic class vanishes, every principal G-bundle on M should have a lift to a 2-bundle $\mathcal{P}$ on M with structure 2-group $\mathcal{P}_k G$. It is tempting to conjecture that the geometry of these 2-bundles is closely related to the enriched elliptic objects of Stolz and Teichner [20].

10.7 Strict 3-Groups

We end this section on 2-groups and the 2-group $\mathcal{P}_k G$ with a brief remark on strict 3-groups and a 3-group extension of $\mathcal{P}_k G$ (which is not from [32]). This thread will then be taken up again in a remark on 3-bundles in §12.3 (p.313) which serves to substantiate the statement from the last section §10.6 (p.235), that $\mathcal{P}_k G$-2-bundles should be obstructed by the first Pontryagin class.

10.7.1 Strict 3-Groups and 2-Crossed Modules

Let $\mathcal{C}$ be any 2-category. Denote composition of 1-morphisms by $\circ_1$ and composition of 2-morphisms by $\circ_2$. A strict 2-functor on $\mathcal{C}$ respects both of these compositions strictly.

A **strict 3-group** $\mathcal{G}$ is a strict 2-category with a strict 2-functor

$$\cdot : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$$

which satisfies the axioms of a group product strictly.

There are other, equivalent, definitions of strict 3-groups. For instance a strict 3-group is (I believe) also the same as a strict 3-category with a single object and all 1-, 2-, and 3-morphisms invertible.

In any case, there are several exchange laws for strict 3-groups. In the formulation using a product 2-functor, which is one one I will use here, two exchange laws comes from the 2-functoriality of '$\cdot$'. Another one is the exchange between $\circ_1$ and $\circ_2$ in a strict 2-category. But only two of these three laws turn out to be independent.

When we forget about the 2-morphisms in $\mathcal{G}$ we obtain a strict 2-group with objects and 1-morphisms those of $\mathcal{G}$. This 2-group is given by a crossed module (G, H, α_1, t_1) where G and H are any groups and

$$\begin{aligned} \alpha_1 &: G \to \mathrm{Aut}(H) \\ t_1 &: H \to G \end{aligned}$$

are homomorphisms.

Recall that under the product operation '$\cdot$' the 1-morphisms of $\mathcal{G}$ form the group $G \rtimes H$ with

$$(g,h)(g',h') = (gg', h\,\alpha(g)\big(h'\big))\,.$$

Alternatively, when we forget about the objects in $\mathcal{G}$ we get another strict 2-group (G', J, α_2, t_2)

$$\begin{aligned} \alpha_2 &: G' \to \mathrm{Aut}(J) \\ t_2 &: J \to G' \end{aligned}$$

whose objects are the 1-morphisms and whose 1-morphisms are the 2-morphisms of $\mathcal{G}$. Hence G' must be the group of 1-morphisms under $\cdot$, i.e.

$$G' = G \rtimes H\,.$$

Collecting all this data we can describe the 3-group $\mathcal{G}$ by a tuple

$$(G, H, J, \alpha_1, \alpha_2, t_1, t_2)$$

There are certain conditions imposed on this data coming from the two versions of the exchange law in $\mathcal{G}$:

Claim:

1. $\mathrm{Im}(t_2) = (1, H) \simeq H \subset G \rtimes H$
2. t_1, t_2 form a sequence (not necessarily exact)

$$J \xrightarrow{t_2} H \xrightarrow{t_1} G\,,$$

   i.e.

$$t_1(t_2(j)) = 1\,.$$

3. J is necessarily an *abelian* group.

When all conditions are satisfied we should call the tuple (G, H, J, α_i, t_i) a **2-crossed module**.

So, to summarize, a 2-crossed module is a tuple

$$(G, H, J, \alpha_1, \alpha_2, t_1, t_2)$$

where G and H are any groups and J is an abelian group, and where

$$\begin{aligned} \alpha_1 &: G \to \mathrm{Aut}(H) \\ \alpha_2 &: G \rtimes H \to \mathrm{Aut}(J) \\ t_1 &: H \to G \\ t_2 &: J \to H \subset G \rtimes H \end{aligned}$$

are homomorphisms such that we have a sequence

$$J \xrightarrow{t_2} H \xrightarrow{t_1} G\,.$$

and such that (G, H, α_1, t_1) and $(G \rtimes H, J, \alpha_2, t_2)$ are two ordinary crossed modules.

Proof.

First some notation:

We label 1-morphisms in the 3-group as usual by their source object $g \in G$ and an element of $h \in H$ as (g, h). Similarly, 2-morphisms are labeled by their source $(g, h) \in G \rtimes H$ and an element $j \in J$ as

$$((g,h), j) : (g,h) \xrightarrow{j} \big((g,h') = t_2(j)\,(g,h)\big)\;.$$

Here source and target are 1-morphisms

$$\begin{aligned} (g,h) : g &\xrightarrow{h} \big(g' = t_1(h)\,g\big) \\ (g,h') : g &\xrightarrow{h'} \big(g' = t_1(h')\,g\big) \end{aligned}$$

which must share the same source and target objects $g, g' \in G$.

It follows that for h and h' here we have

$$t_1(h) = t_1(h') \ . \tag{10.17}$$

When plugged into the equation

$$(g, h') = t_2(j)\,(g, h)$$

for the target of $((g, h), j)$ one gets

$$\begin{aligned} t_2(j) &= (g, h')(g, h)^{-1} \\ &= (g, h')(g^{-1}, \alpha_1(g^{-1})(h^{-1})) \\ &= (1, h'h^{-1})\,. \end{aligned}$$

This proves first of all that the image of t_2 is $(1, H) \subset G \rtimes H$.

So if we identify this image with H

$$(1, h'h^{-1}) \simeq h'h^{-1}$$

and apply $t_1 : H \to G$ to it, we get

$$\begin{aligned} t_1(t_2(j)) &= t_1(h'h^{-1}) \\ &= t_1(h')\,t_1(h)^{-1} \\ &\overset{(10.17)}{=} 1\,. \end{aligned}$$

This shows that we have a sequence $J \xrightarrow{t_2} H \xrightarrow{t_1} G$.

The above facts were consequences of the exchange laws in the 2-groups $(G, H, \alpha_1, \alpha_2)$ and (G', J, α_2, t_2) that sit inside the 3-group $\mathcal{G}$. The exchange law in (G', J, α_2, t_2) comes from the functoriality of '$\cdot$' with respect to $\circ_2$. The exchange law in $(G, H, \alpha_1, \alpha_2)$ comes from functoriality of '$\cdot$' with respect to $\circ_1$ restricted to identity 2-morphisms. When this is generalized to non-identity 2-morphisms one obtains a further condition:

Consider two composable 1-morphisms (g, h) and $(t_1(h)\,g, h')$ with composition

$$(g, h) \circ_1 (t_1(h)\,g, h') = (g, h'h)\,.$$

Now let $((g, h), j)$ and $((t_1(h)\,g, h'), j')$ be two 2-morphisms with source (g, h) and $(t_1(h)\,g, h')$, respectively.

Lemma: Under the composition $\circ_1$ the 2-morphisms behave as

$$((g, h), j) \circ_1 \left((t_1(h)\,g, h'), j'\right) = \left((g, h'h),\ j'\,\alpha_2(1, h')(j)\ .\right)$$

Proof of the lemma:

First of all we can identically write:

$$((g, h), j) \circ_1 \left((t_1(h)\,g, h'), j'\right) = [((1, 1), 1) \cdot ((g, h), j)] \circ_1 \left[((1, h'), j') \cdot ((t_1(h)\,g, 1), 1)\right]$$

Using the exchange law (2-functorality of '·') this becomes

$$\cdots = \big[((1,1),1) \circ_1 ((1,h'),j')\big] \cdot [((g,h),j) \circ_1 ((t_1(h)\,g,1),1)] \;.$$

Note how now the two $\circ_1$-compositions that appear each involve an *identity* 2-morphism. The axioms for 2-categories say that the identites under $\circ_2$-composition are also identites for $\circ_1$-composition. It follows that

$$\begin{aligned}\cdots &= ((1,h'),j') \cdot ((g,h),j) \\ &= ((g,h'h), j'\,\alpha_2\big(1,h'\big)(j))\,.\end{aligned}$$

This poves the above lemma. $\square$

Noting that the $\circ_1$-product is hence once again a kind of semidirect product one sees that the lemma can equivalently be obtained by studying the exchange law between $\circ_1$ and $\circ_2$.

With this product rule in hand we can now do a variation of the Eckmann-Hilton argument:

Consider the expression

$$[((1,1),j) \circ_1 ((1,1),1)] \cdot \big[((1,1),1) \circ_1 ((1,1),j')\big] \;.$$

Evaluating this as indicated yields

$$\cdots = ((1,1), jj')\,.$$

Using the exchange law first gives

$$\cdots = [((1,1),j) \cdot ((1,1),1)] \circ_1 \big[((1,1),1) \cdot ((1,1),j')\big]$$

and using the above lemma this becomes

$$\cdots = ((1,1), j'j)\,.$$

This shows that J must be an abelian group. $\square$

10.7.2 A 3-group extension of $\mathcal{P}_k G$

Given any crossed module (G,H,α_1,t_1) we may ask what extensions (G,H,J,α_i,t_i) to a 2-crossed module there are.

By the above result this amounts to finding an abelian group J and homomorphisms $\alpha_2 : G \rtimes H \to \mathrm{Aut}(J)$ and $t_2 : J \to G \rtimes H$ such that

$$t_1(t_2(j)) = 1\,, \qquad \forall\, j \in J\,.$$

Consider the strict 2-group

$$\mathcal{P}_k G = (P_0 G, \widehat{\Omega_k G}, \alpha_1, t_1)$$

from above. There is an obvious choice for the above extension:

Let $J = U(1)$ be the abelian group and identify this by

$$\begin{aligned} t_2 : J &\to \widehat{\Omega_k G} \\ j &\mapsto (1, j) \end{aligned}$$

with the center of $H = \widehat{\Omega_k G}$. This gives an exact sequence

$$J = U(1) \xrightarrow{t_2} \widehat{\Omega_k G} \xrightarrow{t_1} P_0 G$$

and hence this t_2 is admissible.

11. 2-Connections with 2-Holonomy on 2-Bundles

Bartels [36] defined a concept of '2-bundle' by categorifying the usual concept of bundle. Originally his definition only treated 2-bundles over an ordinary space. However, to handle twisted nonabelian gerbes, we need 2-bundles over a 2-space of infinitesimal loops. So, we define locally trivial 2-bundles over an arbitrary simple 2-space in §11.1 (p.242), describe them using local data in §11.2 (p.244), and specialize to the case where the base space consists of infinitesimal loops in §11.2.4 (p.251).

11.1 Locally Trivial 2-Bundles

In differential geometry an ordinary bundle consists of two smooth spaces, the **total space** E and the **base space** B, together with a **projection map**

$$E \xrightarrow{p} B\,.$$

To categorify the theory of bundles, we start by replacing smooth spaces by smooth 2-spaces:

Definition 11.1. *A* **2-bundle** *consists of*

- *a 2-space P (the* **total 2-space***)*
- *a 2-space B (the* **base 2-space***)*
- *a smooth map $p\colon P \to B$ (the* **projection***) .*

In gauge theory we are interested in *locally trivial* 2-bundles. Ordinarily, a locally trivial bundle with fiber F is a bundle $E \xrightarrow{p} B$ together with an open cover U_i of B, such that the restriction of E to any of the U_i is equipped with an isomorphism to the trivial bundle $U_i \times F \to U_i$. To categorify this, we need to define a '2-cover' of the base 2-space B.

Definition 11.2. *A 2-space S is called a* **sub-2-space** *of the 2-space B if $\mathrm{Ob}(S) \subseteq \mathrm{Ob}(B)$, $\mathrm{Mor}(S) \subseteq \mathrm{Mor}(B)$, and the source, target, identity and composition maps of S are the restriction of these maps on B to these subsets. In this case we write $S \subseteq B$.*

Definition 11.3. *If S is a sub-2-space of B, $\mathrm{Ob}(S)$ is open in $\mathrm{Ob}(B)$ and $\mathrm{Mor}(S)$ is open in $\mathrm{Mor}(B)$, we say S is an* **open** *sub-2-space of B.*

Definition 11.4. *Given a simple 2-space B ((def. 9.10)) and a collection of sub-2-spaces $\{U_i\}_{i\in I}$ with $U_i \subseteq B$, their* **union** *$\bigcup_{i\in I} U_i$ is defined to be the sub-2-space of B whose space of objects is $\bigcup_{i\in I} \mathrm{Ob}(U_i)$ and whose space of morphisms is $\bigcup_{i\in I} \mathrm{Mor}(U_i)$. Similarly, the* **intersection** *$\bigcap_{i\in I} U_i$ is defined to be the sub-2-space of B whose space of objects is $\bigcap_{i\in I} \mathrm{Ob}(U_i)$ and whose space of morphisms is $\bigcap_{i\in I} \mathrm{Mor}(U_i)$.*

We restrict this definition of 'union' to simple 2-spaces B to avoid situations where one can compose a morphism $f\colon x \to y$ in U_{i_1} with a morphism $g\colon y \to z$ in U_{i_2} to obtain a morphism that lies in none of the U_i. This happens, for example, when B is the path groupoid of some space ((def. 9.3)). Such 2-spaces require a more general concept of '2-cover' than we provide here.

Definition 11.5. *A* **2-cover** *of a simple 2-space B is a collection $\{U_i\}_{i\in I}$ of open sub-2-spaces of B with $\bigcup\limits_{i\in I} U_i = B$.*

Given a 2-cover $\{U_i\}_{i\in I}$ of B, we can form a 2-space called their **disjoint union** $U = \coprod\limits_{i\in I} U_i$ in an obvious way, and every 2-cover is equipped with a 2-map

$$U \xrightarrow{j} B \tag{11.1}$$

that restricts on each U_i to the inclusion $U_i \hookrightarrow B$. We often refer to the 2-cover $\{U_i\}_{i\in I}$ simply as U for short.

Definition 11.6. *Given a 2-cover U of a simple 2-space B, we define the* **2-space of n-fold intersections** *by:*

$$U^{[n]} = \coprod_{i_1,i_2,\ldots,i_n\in I} U_{i_1}\cap U_{i_2}\cap\cdots\cap U_{i_n}\,.$$

This 2-space comes with maps

$$\begin{array}{ccc} U^{[n]} & \xrightarrow{j_{01\cdots(k-1)(k+1)\cdots n}} & U^{[n-1]} \\ (i_1,i_2,\ldots,i_n,x\xrightarrow{\gamma}y) & \mapsto & (i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_n,x\xrightarrow{\gamma}y) \end{array} \tag{11.2}$$

that forget about the kth member of the multiple intersection.

With the notion of 2-cover in hand, we can now state the definition of a locally trivial 2-bundle. First note that we can restrict a 2-bundle $E\xrightarrow{p}B$ to any sub-2-space $U\subseteq B$ to obtain a 2-bundle which we denote by $P|_U \xrightarrow{p} U$. Then:

Definition 11.7. *Given a 2-space F, we define a* **locally trivial 2-bundle with fiber** *F to be a 2-bundle $E\xrightarrow{p}B$ ((def. 11.1)) and a 2-cover U of the base 2-space B equipped with equivalences ((def. 9.11))*

$$P|_{U_i} \xrightarrow{t_i} U_i\times F$$

such that the diagrams

$$\begin{tikzcd} P|_{U_i} \arrow[rr, "t_i"] \arrow[dr, "p"'] & & U_i\times F \arrow[dl] \\ & U_i & \end{tikzcd}$$

commute up to invertible 2-maps for all $i\in I$.

This definition is concise and elegant, but rather abstract. In §11.2 (p.244) we translate its meaning into transition laws for local data specifying the 2-bundle. In order to do so, we first need to extract *transition functions* from a local trivialization:

By composing the local trivializations and their weak inverses on double intersections U_{ij} one gets autoequivalences of $U_{ij} \times F$ of the form

$$U_{ij} \times F \xrightarrow{\bar{t}_i \circ t_j} U_{ij} \times F$$

and similarly for other index combinations.

Definition 11.8. *Given a 2-bundle equipped with a local trivialization such that*

- *all autoequivalences* $U_{ij} \times F \xrightarrow{\bar{t}_i \circ t_j} U_{ij} \times F$ *act trivially on the* U_{ij} *factor, so that*

$$\bar{t}_i \circ t_j = \mathrm{id}_{U_{ij}} \times g_{ij} \,,$$

- F *is a 2-group* $\mathcal{G}$*,*

- *the* g_{ij} *act by left horizontal 2-group multiplication on* F

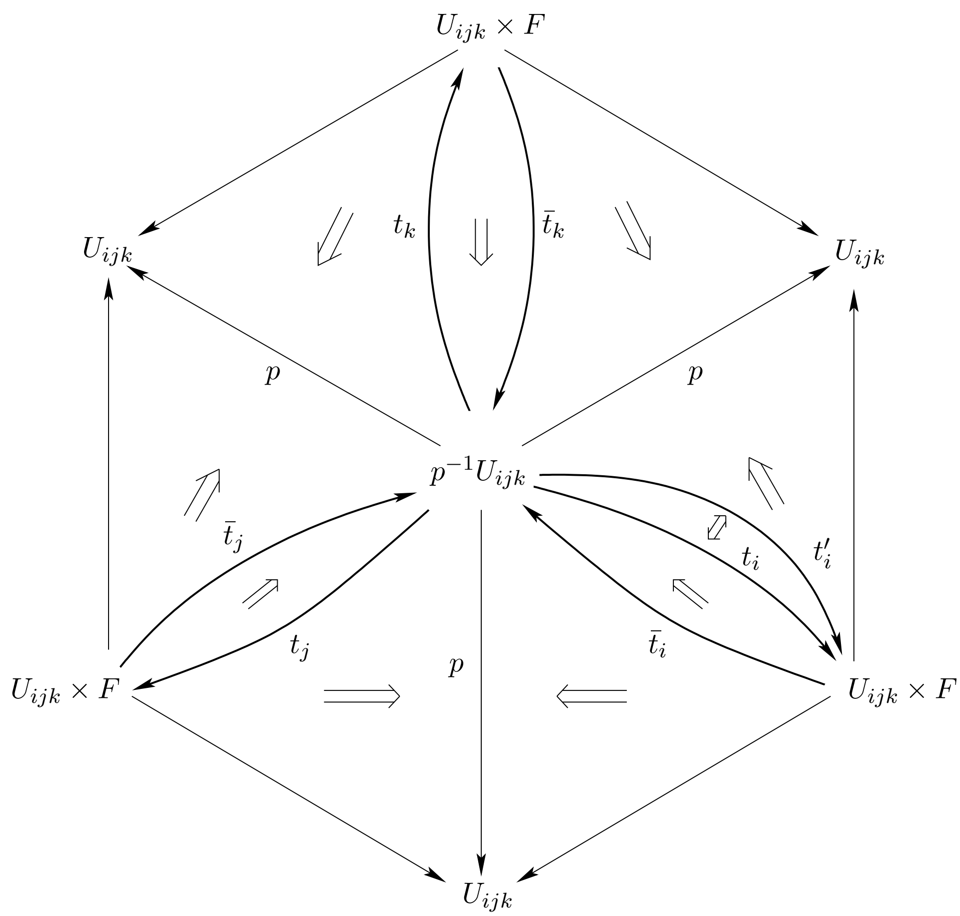
we say that our 2-bundle is a **principal $\mathcal{G}$-2-bundle** *and that*

$$\begin{array}{ccc} U^{[2]} & \xrightarrow{g} & \mathcal{G} \\ U_{ij} & \mapsto & g_{ij} \end{array}$$

is the **transition function***.*

11.2 2-Transitions in Terms of Local Data

Consider a triple intersection $U_{ijk} = U_i \cap U_j \cap U_k$ in a principal $\mathcal{G}$-2-bundle ((def. 11.8)). The existence of the local trivialization implies that the following diagram 2-commutes (all morphisms here are 2-maps and all 2-morphisms are natural isomorphisms between these):



Compared to the analogous diagram for an uncategorified bundle two important new aspects are that the barred morphisms are inverses-up-to-isomorphism of the local trivializations and that the local trivialization itself is unique only up to natural isomorphisms ((def. 11.7)). The latter is indicated by the presence of an arrow denoting a trivialization t'_i naturally isomorphic to t_i.

From the diagram it is clear that the usual transition law $g_{ij}g_{jk} = g_{ik}$ here becomes a natural isomorphism called a 2-transition, which was first considered in [36] for the special case of trivial base 2-spaces, but which directly generalizes to arbitrary base 2-spaces:

Definition 11.9. *Given a base 2-space B with cover $U \xrightarrow{j} B$ a* **2-transition** *is*

- *a 2-map*

$$U^{[2]} \xrightarrow{g} \mathcal{G}$$

called the **transition function***,*

- *and a natural isomorphism* f

$$\begin{array}{c} U^{[3]} \xrightarrow{j_{02}} U^{[2]} \xrightarrow{g} \mathcal{G} \\ \stackrel{f}{\Longrightarrow} \\ U^{[3]} \xrightarrow{\overset{\vee}{U}^{[3]}} U^{[3]} \times U^{[3]} \xrightarrow{j_{01}\times j_{12}} U^{[2]} \times U^{[2]} \xrightarrow{g\times g} \mathcal{G} \times \mathcal{G} \xrightarrow{m} \mathcal{G}\,, \end{array}$$

  (which expressed the categorification of the ordinary transition law $g_{ij}g_{jk} = g_{ik}$*), together with the coherence law for* f *enforcing the associativity of the product* $g_{ij}g_{jk}g_{kl}$*,*

- *and a natural isomorphism*

$$\begin{array}{c} U \xrightarrow{j_{00}} U^{[2]} \xrightarrow{g} \mathcal{G} \\ \stackrel{\eta}{\Longrightarrow} \\ U \xrightarrow{\hat{U}} 1 \xrightarrow{i} \mathcal{G}\,. \end{array}$$

  (expressing the categorification of the ordinary $g_{ii} = 1$*) together with its coherence laws.*

In the above $U^{[3]} \xrightarrow{\overset{\vee}{U}^{[3]}} U^{[3]} \times U^{[3]}$ denotes the diagonal embedding of $U^{[3]}$ in its second tensor power and m denotes the horizontal multiplication (functor) in the 2-group $\mathcal{G}$. The maps $j_{\dots}$ have been defined in (11.2).

In terms of local functions this means the following:

Proposition 11.1. *A 2-transition ((def. 11.9)) on a* $\mathcal{G}$*-2-bundle with base 2-space being a simple 2-space ((def. 9.10)) and* $\mathcal{G}$ *a 2-group induces the transition law (9.21) of a nonabelian gerbe.*

Proof.

The existence of the natural isomorphism means that there is a map

$$\begin{aligned} (U^{[3]})^1 &\xrightarrow{f} \mathcal{G}^2 \\ (x,i,j,k) &\mapsto h_{ijk}(x)\ , \end{aligned}$$

with the property

$$g^2_{ik}(x) \circ h_{ijk}(x) = h_{ijk}(x) \circ (g^2_{ij}(x)\cdot g^2_{jk}(x))\,, \qquad \forall\,(x,i,j) \in U^{[2]}\,. \tag{11.3}$$

(Here $\circ$ denotes the vertical and $\cdot$ the horizontal product in the 2-group, see Prop. 9.1)

For $\mathcal{G}$ the source/target matching condition implies that (again Prop. 9.1)

$$t\big(f^2_{ijk}\big)\, g^1_{ik} = g^1_{ij} g^1_{jk}\,, \tag{11.4}$$

where we have decomposed the 2-group element

$$h_{ijk}(x) = (f^1_{ijk}(x)\,, f^2_{ijk}(x))$$

into its source label $f^1_{ijk}(x) \in G$ and its morphism label $f^2_{ijk}(x) \in H$.

Identifying $g_{ij} = \phi_{ij}$ this is the gerbe transition law (9.21). $\square$

11.2.1 2-Transitions and Cocycles

The functor that glues different local trivializations of the 2-bundles is

$$\bar{t}_i \circ t_j : U_{ij} \times \mathcal{G} \to U_{ij} \times \mathcal{G} \, .$$

Let us concentrate on the case where these transitions act trivially on the U_{ij} factor and act by left multiplication on $\mathcal{G}$.

A little thought shows that this means that the $\bar{t}_i \circ t_j$ must come from a functor

$$g_{ij} : U_{ij} \to \mathcal{G} \, ,$$

which assigns a morphisms $g_{ij}(x) \xrightarrow{g_{ij}(\gamma)} g'_{ij}(x)$ in the 2-group $\mathcal{G}$ to every morphism $x \xrightarrow{\gamma} y$ in U_{ij}, as follows:

$$\bar{t}_i \circ t_j \Big(x \xrightarrow{\gamma} y, f \Big) = \Big(x \xrightarrow{\gamma} y, \; g_{ij}(\gamma) \cdot f \Big) \, .$$

Here $f \in \mathcal{G}$.

The usual equation

$$g_{ij} g_{jk} = g_{ik}$$

describing the consistency of transitions in an ordinary principal 2-bundle now becomes a natural isomorphism between functors, whose naturality square is the following:

$$
\begin{array}{c|ccc}
U_{ijk} & \mathcal{G} & & \mathcal{G} \\ \hline
x & g_{ik}(x) & \xrightarrow{f_{ijk}(x)} & g_{ij}(x) \cdot g_{jk}(x) \\
\gamma \big\downarrow & g_{ik}(\gamma) \big\downarrow & & g_{ij}(\gamma) \cdot g_{jk}(\gamma) \big\downarrow \\
y & g_{ik}(y) & \xrightarrow{f_{ijk}(y)} & g_{ij}(y) \cdot g_{jk}(y)
\end{array}
$$

The morphisms

$$g_{ik}(x) \xrightarrow{f_{ijk}(x)} g_{ij}(x) \cdot g_{jk}(x)$$

in $\mathcal{G}$ define the natural transformation. When we write the morphism $f_{ijk}(x)$ in terms of elements of $G \ltimes H$ as

$$h_{ijk}(x) = (g_{ij}(x) \, g_{jk}(x) \, , \; f_{ijk}(x))$$

the existence of these arrows is expressed by the equation

$$g_{ij}(x) \, g_{jk}(x) = t(f_{ijk}(x)) \, g_{ik}(x) \, .$$

This is the first cocylce relation.

It is accompanied by another equation, which is not called a cocycle relation, but which appears here notwithstanding. It is the one expressing the commutativity of the above naturality square and reads

$$f_{ijk}(y) \, g_{ik}(\gamma) \, f_{ijk}^{-1}(x) = h_{ij}(\gamma) \, \alpha(g_{ij}(x))(h_{jk}(\gamma)) \, ,$$

where we have written $g(\gamma)$ in terms of $(g, h) \in G \ltimes H$.

11.2.2 Interpretation in terms of transition bundles

In order to grasp the meaning of this equation it may be instructive to compare the above to the analogous step in the context of bundle gerbes, as detailed in [50] (where the last equation above corresponds to equation (106)).

In that context one declares that instead of transition *functions* one is to use transition *bundles* E which are principal H-bibundles.

So on the fibers of such a bundle the group H acts freely and transitively from the left

$$\begin{aligned} H \ni h : E &\to E \\ e &\mapsto he \end{aligned}$$

and from the right $e \mapsto eh$. Being both free and transitive, these two actions must be related by an automorphism $\phi_e \in \mathrm{Aut}(H)$ as

$$eh \equiv \phi_e(h)\, e\,.$$

A local trivializaton of such a bundle consists of choosing a good cover $\{O_\alpha\}$ and local sections $\sigma^\alpha : O_\alpha \to E$. These give rise to the local maps

$$\phi^\alpha \equiv \sigma^\alpha \circ \phi$$

and on overlapping patches they are related by transition functions $h\alpha\beta : O_{\alpha\beta} \to H$ as

$$\sigma_\alpha = h^{\alpha\beta}\sigma_\beta\,.$$

For bibundles E_1 and E_2 one can form the product bundle E_1E_2 defined fiberwise over H. So if E_1 is trivialized by $\{\sigma^1_\alpha\}$ with local data $\left\{h_1^{\alpha\beta}, \phi_1^\alpha\right\}$ and E_2 by $\{\sigma_2^\alpha\}$ with local data $\left\{h_2^{\alpha\beta}, \phi_2^\alpha\right\}$, then the product bundle is trivialized by sections $\{(\sigma_1^\alpha, \sigma_2^\alpha)\}$ such that the transitions are given by

$$\begin{aligned} (\sigma_1^\alpha, \sigma_2^\alpha) &= (h_1^{\alpha\beta}\sigma_1^\beta, h_2^{\alpha\beta}\sigma_2^\beta) \\ &= h_1^{\alpha\beta}(\sigma_1^\beta h_2^{\alpha\beta}, \sigma_2^\beta) \\ &= h_1^{\alpha\beta}\phi_1^\beta\left(h_2^{\alpha\beta}\right)(\sigma_1^\beta, \sigma_2^\beta)\,. \end{aligned}$$

The maps ϕ of the product bundle are similarly seen to be given by $\phi_1^i\phi_2^i$. Hence the product bundle is given by the local data

$$\left\{h_1^{\alpha\beta}\phi_1^\beta\left(h_2^{\alpha\beta}\right), \phi_1^\alpha\phi_2^\beta\,.\right\}$$

If a bibundle is trivial, it admits a global section represented by $f^\alpha : O_\alpha \to H$ and its local data in the above sense reads $\left\{f^\alpha(f^\beta)^{-1}, \mathrm{Ad}(f^\alpha)\right\}$.

If E_1 is described by $\{\sigma_i^1\}$ with $\left\{h_{ij}^1, \phi_i^1\right\}$ and $\{\sigma_i^2\}$ with E_2 by $\left\{h_{ij}^2, \phi_i^2\right\}$, then their product, which is locally trivialized by the sections (σ_i^1, σ_i^2) is described by

$$\{h_{ij}^1, \phi_i^1\} \cdot \{h_{ij}^2, \phi_i^2\} \equiv \{h_{ij}^1\}\,.$$

Now, in the context of bundle gerbes one chooses a good cover $\{U_i\}$ of base space M and gets *transition bi-bbundles* $\{E_{ij} \to U_{ij}\}_{i,j}$ over the double intersections. The transition law now says that on triple intersections

$$E_{ij}E_{jk} = T_{ijk}E_{ik} \,,$$

where T_{ijk} is a bundle of the above form with

$$T_{ijk} \simeq \left\{ f^\alpha_{ijk}(f^\beta_{ijk})^{-1}, \mathrm{Ad}_{f^\alpha_{ijk}} \right\} .$$

By the above formulas this yields the following two conditions on the local data of these bundles:

$$\phi^\alpha_{ij}\phi^\beta_{ij} = \mathrm{Ad}_{f^\alpha_{ijk}}\phi^\alpha_{ik}$$

and

$$h^{\alpha\beta}_{ij}\phi^\beta_{ij}\left(h^{\alpha\beta}_{ij}\right) = f^\alpha_{ijk}h^{\alpha\beta}_{ik}(f^\beta_{ijk})^{-1} .$$

The first of these corresponds to the existence of the arrows in the natural transformation discussed above. The second corresponds to the commutativity of the naturality square.

The full discussion of these nonabelian bundle gerbes requires the consideration of bundles over fiver products of fibrations of base space. The interested reader is referred to [50] for further details.

11.2.3 The coherence law for the 2-transition

The natural transformation f which weakens the ordinary transition law $g_{ij}g_{jk} = g_{ik}$ has to satisfy a coherence law which makes its application on multiple products $g_{ij}g_{jk}g_{kl}$ well defined.

Note that first of all that Prop. (9.1) implies a certain relation among the h_{ijk}: By using the relation $g^1_{ij}g^1_{jk} = t(h_{ijk})\, g^1_{ik}$ in the expression $g_{ij}g_{jk}g_{kl}$ in two different ways one obtains

$$t(h_{ijk})\, t(h_{ikl}) = g_{ij}t(h_{jkl})\, g^{-1}_{ij}t(h_{ijl}) .$$

This equation implies that

$$f^{-1}_{ikl}\, f^{-1}_{ijk}\alpha(g_{ij})(h_{jkl})\, h_{ijl} = \lambda_{ijkl} \tag{11.5}$$

with

$$\lambda_{ijkl} \colon U^1_{ijkl} \to \ker(t) \subset H .$$

This is the gerbe transition law (9.26). The function λ_{ijkl} is the ‘twist’ 0-form (9.17).

From the perspective of 2-bundles the twist can be understood as coming from a nontrivial natural transformation between 2-maps from $U^{[4]}$ to $U^{[2]}$:

First assume that the natural transformation

$$\begin{array}{c} U^{[4]} \overset{j_{023}\circ j_{02}}{\longrightarrow} U^{[2]} \\ \overset{\omega_{03}}{\Rightarrow} \\ U^{[4]} \overset{j_{013}\circ j_{02}}{\longrightarrow} U^{[2]} \,. \end{array} \tag{11.6}$$

is trivial, which means that sending a based loop $\gamma_{(x,i,j,k,l)}$ in $(U^{[4]})^2$ first to the based loop $\gamma_{(x,i,k,l)}$ in $(U^{[3]})^2$ and then to $\gamma_{(x,i,l)}$ in $(U^{[2]})^2$ yields the same result as first sending it to $\gamma_{(x,i,j,l)}$ in $(U^{[3]})^2$ and then to $\gamma_{(x,i,l)}$ in $(U^{[2]})^2$.

Using (11.3) we have

$$\begin{aligned}
& (g^2_{ij}\cdot g^2_{jk})\cdot g^2_{kl} = g^2_{ij}\cdot(g^2_{jk}\cdot g^2_{kl}) \\
\overset{(11.3)}{\Leftrightarrow}\; & ((h_{ijk})^r\circ g^2_{ik}\circ h_{ijk})\cdot\left(1_{g^1_{kl}}\circ g^2_{kl}\circ 1_{g^1_{kl}}\right) = \left(1_{g^1_{ij}}\circ g^2_{ij}\circ 1_{g^1_{ij}}\right)\cdot((h_{jkl})^r\circ g^2_{jl}\circ h_{jkl}) \\
\Leftrightarrow\; & \left((h_{ijk})^r\cdot 1_{g^1_{kl}}\right)\circ(g^2_{ik}\cdot g^2_{kl})\circ\left(h_{ijk}\cdot 1_{g^1_{kl}}\right) = \left(1_{g^1_{ij}}\cdot(h_{jkl})^r\right)\circ(g^2_{ij}\cdot g^2_{jl})\circ\left(1_{g^1_{ij}}\cdot h_{jkl}\right) \\
\overset{(11.3)}{\Leftrightarrow}\; & \left((h_{ijk})^r\cdot 1_{g^1_{kl}}\right)\circ((h_{ikl})^r\circ g^2_{il}\circ h_{ikl})\circ\left(h_{ijk}\cdot 1_{g^1_{kl}}\right) = \left(1_{g^1_{ij}}\cdot(h_{jkl})^r\right)\circ((h_{ijl})^r\circ g^2_{il}\circ h^2_{ijl})\circ\left(1_{g^1_{ij}}\cdot h_{jkl}\right).
\end{aligned} \tag{11.7}$$

This has the form

$$A^r\circ g^2_{il}\circ A = B^r\circ g^2_{il}\circ B$$

with

$$\begin{aligned} A &= h_{ikl}\circ(h_{ijk}\cdot 1_{g^1_{kl}}) \\ B &= h_{ijl}\circ(1_{g^1_{ij}}\cdot h_{jkl})\,. \end{aligned} \tag{11.8}$$

If we identify both 'conjugations' we obtain

$$A = B \;\Leftrightarrow\; (f^2_{ikl})^{-1}(f^2_{ijk})^{-1}\alpha\big(g^1_{ij}\big)\big(f^2_{jkl}\big)\,f^2_{ijl} = 1\,. \tag{11.9}$$

This reproduces (11.5) without the twist.

Now generalize to nontrivial natural transformations (11.6). This implies the existence of a function

$$(U^{[4]})^1 \xrightarrow{\ell} (U^{[2]})^2$$

that assigns loops based in double overlaps to points in quadruple overlaps. Applying the 'transition function' g to these loops implies that

$$g^2(\ell)\circ g^2(j_{013}\circ j_{02}) = g^2(j_{023}\circ j_{02})\circ g^2(\ell)\;. \tag{11.10}$$

The 2-group element $g^2(\ell)$ is specified by a function

$$(U^{[4]})^1 \xrightarrow{\lambda} \ker(t)\subset H$$

as

$$\begin{aligned} (U^{[4]})^1 &\xrightarrow{\ell\circ g} \mathcal{G}^2 \\ (x,i,j,k,l) &\mapsto (g^1_{il},\lambda_{ijkl}(x))\,. \end{aligned}$$

All this applies to (11.7) by noting that there on the left hand side the g_{il} in general is $g(j_{023}\circ j_{02})$ while that on the right hand is $g(j_{013}\circ j_{02})$.

Hence in the case of nontrivial arrow base space we have to replace the g^2_{il} in the last line on the left with $g^2(j_{013} \circ j_{02})$ and that on the right with $g^2(\ell) \circ g^2(j_{013} \circ j_{02}) \circ (g^2(\ell))^{-1}$.

When doing so the 2-group elements A and B of (11.8) become

$$
\begin{aligned}
A &= (g^1_{il}, \lambda^{-1}_{ijkl}) \circ h_{ikl} \circ (h_{ijk} \cdot 1_{g^1_{kl}}) \\
B &= h_{ijl} \circ (1_{g^1_{ij}} \cdot h_{jkl}) \,.
\end{aligned}
$$

Equating these generalizes (11.9) to

$$
A = B \;\Leftrightarrow\; (f^2_{ikl})^{-1}(f^2_{ijk})^{-1}\alpha\big(g^1_{ij}\big)\big(f^2_{jkl}\big)\, f^2_{ijl} = \lambda_{ijkl} \,.
$$

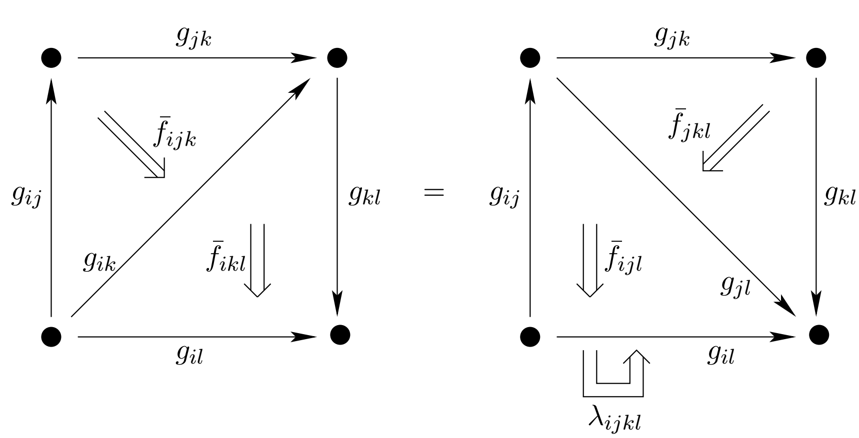
11.2.4 Restriction to the case of trivial base 2-space

It is instructive to restrict the above general disucssion to the case where the base 2-space is trivial:

In that case the 2-transition specifies the following data:

- smooth maps
$$
g_{ij} \colon U_i \cap U_j \to \mathcal{G}^1
$$
- smooth maps
$$
h_{ijk} \colon U_i \cap U_j \cap U_k \to \mathcal{G}^2
$$
with
$$
h_{ijk}(x) : g_{ik}(x) \to g_{ij}(x)\, g_{jk}(x)
$$
- smooth maps
$$
k_i : U_i \to \mathcal{G}^2
$$
with
$$
k_i \colon g_{ii} \to 1 \in \mathcal{G} \,.
$$

The coherence law (11.5) says that on quadruple intersections $U_i \cap U_j \cap U_k \cap U_l$ the following 2-morphisms in $\mathcal{G}$ are identical:



This diagram gives a nice visualization of the different ways to go from the upper arc $g_{ij}g_{jk}g_{kl}$ of the square to the bottom edge g_{il}.

There are also coherence laws for k_i, the **left unit law** and **right unit law**, which express the relation of k to f when two of the indices of the latter coincide:

$$
\begin{tikzcd}
& \bullet \arrow[dr, "g_{ij}"] & \\
\bullet \arrow[ur, bend left, "g_{ii}"] \arrow[ur, "1"', "k_i"] \arrow[rr, "g_{ij}"'] & \Downarrow 1 & \bullet
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& \bullet \arrow[dr, "g_{ij}"] & \\
\bullet \arrow[ur, "g_{ii}"] \arrow[rr, "g_{ij}"'] & \Downarrow f^r_{iij} & \bullet
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \bullet \arrow[dr, bend left, "g_{jj}"] \arrow[dr, "1"', "k_j"] & \\
\bullet \arrow[ur, "g_{ij}"] \arrow[rr, "g_{ij}"'] & 1\Downarrow & \bullet
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& \bullet \arrow[dr, "g_{jj}"] & \\
\bullet \arrow[ur, "g_{ij}"] \arrow[rr, "g_{ij}"'] & \Downarrow f^r_{ijj} & \bullet \mathrm{X}
\end{tikzcd}
$$

The freedom of having nontrivial k_i is special to 2-bundles and not known in (non-abelian) gerbe theory. Gerbe cocycles involve Čech cohomology and hence *antisymmetry* in indices $i, j, k, \ldots$ in the sense that group valued functions go into their inverse on an odd permutation of their cover indices.

Whenever we derive nonabelian gerbe cocycles from 2-bundles with 2-connection we will hence have to restrict to $k_i = 1$ for all i.

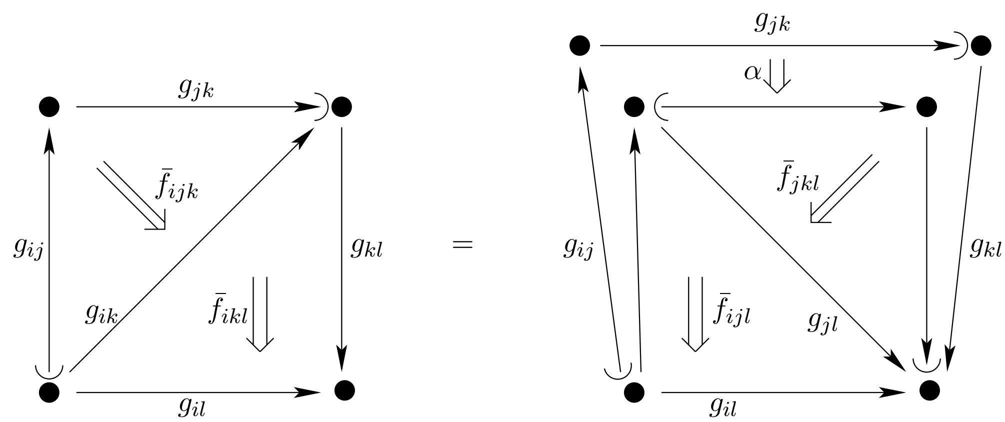
11.2.5 Weak Principal 2-Bundles

The above discussion applied to principal 2-bundles whose structure group is a *strict* 2-group. It can however easily be generalized to coherent weak structure 2-groups:

For weak 2-groups the above coherence law for the transition functions has to take into account that the product of the edge labels is not associative, but that instead there is the **associator**, a morphism

$$(g_{ij} \cdot g_{jk}) \cdot g_{kl} \overset{\alpha}{\Longrightarrow} g_{ij}(g_{jk} \cdot g_{kl})\,.$$

Therfore the coherence law more generally looks like



This however no longer has a simple translation into a formula for group elements.

The above diagrams applied to trivial base 2-spaces. We can also consider nontrivial base 2-spaces, i.e. those with nontrivial base arrow-space. It turns out however that the local description of a 2-bundle coincides with the cocycle data of a nonabelian gerbe only in the limit that the base 2-space morphisms differ 'infinitesimally' from identity morphisms. What this means is explained in the next subsection.

11.2.6 Summary

By internalizing the concept of an ordinary bundle in the 2-category of 2-spaces (which again are categories internalized in Diff, the category of smooth spaces) one obtains a categorified notion of the fiber bundle concept, called 2-bundle, which differs from an ordinary bundle essentially in that what used to be ordinary maps between sets (like the projection map of the bundle) now become (smooth) functors between categories. This adds to the original bundle (at the 'point level' of the 2-space) a dimensional generalization (at the 'arrow level' of the 2-space) of all concepts involved. In addition to providing new 'degrees of freedom' the categorification weakens former notions of equality.

By re-expressing the abstract arrow-theoretic construction of a 2-bundle in terms of concrete local group- and algebra-valued p-forms, we find a generalization of the ordinary transition laws for such local data in an ordinary bundle. Under certain conditions these generalized transition laws coincide with the cocycle data of nonabelian gerbes.

So far all of this pertained to 2-bundles (and nonabelian gerbes) without a notion of connection. For constructing a categorified connection and hence a notion of nonabelian surface holonomy, it is helpful to consider ordinary connections on spaces of paths in a manifold. This is the content of the next section.

11.3 Local 2-Holonomy and Transitions

We now define what a **2-connection** in a 2-bundle over a categorically trivial base 2-space is supposed to be. Then, in §11.4 (p.262) we list the results concerning the expression

of such 2-connections in terms of local p-form data and sketch the proofs. Making these precise requires some technology, which is developed in §11.5 (p.268).

Locally a p-holonomy is nothing but a p-functor from the p-groupoid of p-paths to the structure p-group. For $p = 1, 2$ these p-groupoids are defined in the next subsection §11.3.1. The definition of global 2-holonomy is then given in §11.3.2 (p.257).

11.3.1 p-Path p-Groupoids

Definition 11.10.

The **path groupoid** $\mathcal{P}_1(U)$ *of a manifold* U *is the groupoid for which*

- *objects are points* $x \in U$
- *morphisms with source* $s \in U$ *and target* $t \in U$ *are thin homotopy equivalence classes* $[\gamma]$ *of parametrized paths* $\gamma \in P_s^t(U)$ *(def. 11.16) (p. 269)*

$$
\begin{tikzcd}
x \arrow[r, bend left, "{[\gamma]}"] & y
\end{tikzcd}
$$

- *composition is given by*

$$
\begin{tikzcd}
x \arrow[r, bend left, "{[\gamma_1]}"] & y \arrow[r, bend left, "{[\gamma_2]}"] & z
\end{tikzcd}
=
\begin{tikzcd}
x \arrow[r, bend left, "{[\gamma_1 \circ \gamma_2]}"] & z
\end{tikzcd}
$$

where

$$
\begin{aligned}
\circ \colon P_x^y(U) \times P_y^z(U) &\to P_x^z(U) \\
(\gamma_1, \gamma_2) &\mapsto \gamma_{1,2}
\end{aligned}
$$

with

$$
\gamma_{1,2}(\sigma) \equiv \begin{cases} \gamma_1(2\sigma) & \text{for } 0 \leq \sigma \leq 1/2 \\ \gamma_2(2\sigma - 1) & \text{for } 1/2 \leq \sigma \leq 1 \end{cases} .
$$

Note that taking thin homotopy equivalence classes makes this composition asscociative and invertible.

Just like ordinary holonomy locally is a functor from the groupoid of paths to an ordinary group, 2-holonomy locally is a 2-functor from some 2-groupoid to a 2-group. This 2-groupoid is roughly that consisting of bounded surfaces in U whose horizontal and vertical composition corresponds to the ordinary gluing of bounded surfaces. This heuristic idea is made precise by the following defition of $\mathcal{P}_2(U)$, the **2-groupoid of bigons**.

First of all a bigon is a 'surface with two corners'. More precisely:

Definition 11.11. *Given any manifold* U *a* **parametrized bigon** *in* U *is a smooth map*

$$
\begin{aligned}
\Sigma \colon [0,1]^2 &\to U \\
(\sigma, \tau) &\mapsto \Sigma(\sigma, \tau)
\end{aligned}
\tag{11.11}
$$

with

$$\Sigma(0,\tau) = s \in U$$
$$\Sigma(1,\tau) = t \in U$$

for given $s, t \in U$, which is constant in a neighborhood of $\sigma = 0, 1$ and independent of τ near $\tau = 0, 1$.

Equivalently, a parametrized bigon is a path in path space $P_s^t(U)$ (def. 11.16)

$$\Sigma\colon [0,1] \to P_s^t(U)$$
$$\tau \mapsto \Sigma(\cdot,\tau)\ ,$$

which is constant in a neighborhood of $\tau = 0, 1$. We call s the **source vertex** *of the bigon, t the* **target vertex**, *$\Sigma(\cdot,0)$ the* **source edge** *and $\Sigma(\cdot,1)$ the* **target edge**.

As with paths, the parametrization involved here is ultimately not of interest and should be divided out:

Definition 11.12. *An* **unparametrized bigon** *or simply a* **bigon** *is a thin homotopy equivalence class $[\Sigma]$ of parametrized bigons Σ (def. 11.11).*

More in detail, this means (*cf.* for instance [48] p.26 and [41] p.50) that two parametrized bigons $\Sigma_1, \Sigma_2 \colon [0,1]^2 \to U$ are taken to be equivalent

$$\Sigma_1 \sim \Sigma_2$$

precisely if there exists a smooth map

$$H\colon [0,1]^3 \to U$$

which takes one bigon smoothly into the other while preserving their boundary, i.e. such that

$$\begin{aligned}
H(\sigma,\tau,0) &= \Sigma_1(\sigma,\tau)\\
H(\sigma,\tau,1) &= \Sigma_2(\sigma,\tau)\\
H(\sigma,0,\nu) &= \Sigma_1(\sigma,0) = \Sigma_2(\sigma,0)\\
H(\sigma,1,\nu) &= \Sigma_1(\sigma,1) = \Sigma_2(\sigma,1)\\
H(0,\tau,\nu) &= \Sigma_1(0,\tau) = \Sigma_2(0,\tau)\\
H(1,\tau,\nu) &= \Sigma_1(1,\tau) = \Sigma_2(1,\tau)\ ,
\end{aligned}$$

but which does so in a degenerate fashion, meaning that

$$\mathrm{rank}(dH)(\sigma,\nu,\tau) < 3$$

for all $\sigma, \tau, \nu \in [0,1]$.

These bigons naturally form a coherent 2-groupoid:

Definition 11.13. *The* **path 2-groupoid** $\mathcal{P}_2(U)$ *of a manifold U is the 2-groupoid whose*

- *objects are points $x \in U$*
- *morphisms are paths $\gamma \in P_x^y(U)$*

$$\begin{tikzcd} x \arrow[r, bend left, "\gamma"] & y \end{tikzcd}$$

- *2-morphisms are bigons (def. 11.12) with source edge γ_1 and target edge γ_2*

$$\begin{tikzcd} x \arrow[r, bend left, "\gamma_1"{name=U}] \arrow[r, bend right, "\gamma_2"'{name=D}] & y \arrow[from=U, to=D, Rightarrow, "{[\Sigma_1]}"] \end{tikzcd}$$

and whose composition operations are defined as

-
$$\begin{tikzcd} x \arrow[r, bend left, "\gamma_1"] & y \arrow[r, bend left, "\gamma_2"] & z \end{tikzcd} = \begin{tikzcd} x \arrow[r, bend left, "\gamma_1 \circ \gamma_2"] & z \end{tikzcd}$$

-
$$\begin{tikzcd} x \arrow[r, bend left=60, "\gamma_1"{name=U}] \arrow[r, "\gamma_2"{name=M}] \arrow[r, bend right=60, "\gamma_3"'{name=D}] & y \arrow[from=U, to=M, Rightarrow, "{[\Sigma_1]}"] \arrow[from=M, to=D, Rightarrow, "{[\Sigma_2]}"] \end{tikzcd} = \begin{tikzcd} x \arrow[r, bend left, "\gamma_1"{name=U}] \arrow[r, bend right, "\gamma_3"'{name=D}] & y \arrow[from=U, to=D, Rightarrow, "{[\Sigma_1 \circ \Sigma_2]}"] \end{tikzcd}$$

-
$$\begin{tikzcd} x \arrow[r, bend left, "\gamma_1"{name=U1}] \arrow[r, bend right, "\gamma_1'"'{name=D1}] & y \arrow[r, bend left, "\gamma_2"{name=U2}] \arrow[r, bend right, "\gamma_2'"'{name=D2}] & z \arrow[from=U1, to=D1, Rightarrow, "{[\Sigma_1]}"'] \arrow[from=U2, to=D2, Rightarrow, "{[\Sigma_2]}"'] \end{tikzcd} = \begin{tikzcd} x \arrow[r, bend left, "\gamma_1 \circ \gamma_2"{name=U}] \arrow[r, bend right, "\gamma_1' \circ \gamma_2'"'{name=D}] & z \arrow[from=U, to=D, Rightarrow, "{[\Sigma_1 \cdot \Sigma_2]}"'] \end{tikzcd}$$

where

$$(\gamma_1 \circ \gamma_2)(\sigma) \equiv \begin{cases} \gamma_1(2\sigma) & \textit{for } 0 \le \sigma \le 1/2 \\ \gamma_2(2\sigma - 1) & \textit{for } 1/2 \le \sigma \le 1 \end{cases}$$

$$(\Sigma_1 \circ \Sigma_2)(\sigma, \tau) \equiv \begin{cases} \Sigma_1(\sigma, 2\tau) & \textit{for } 0 \le \tau \le 1/2 \\ \Sigma_2(\sigma, 2\tau - 1) & \textit{for } 1/2 \le \tau \le 1 \end{cases}$$

$$(\Sigma_1 \cdot \Sigma_2)(\sigma, \tau) \equiv \begin{cases} \Sigma_1(2\sigma, \tau) & \textit{for } 0 \le \sigma \le 1/2 \\ \Sigma_2(2\sigma - 1, \tau) & \textit{for } 1/2 \le \sigma \le 1 \end{cases} .$$

Note that in this definition we did *not* divide out by thin homotopy of parametrized *paths* but only by thin homotopy of parametrized bigons. This implies that the horizontal composition in this 2-groupoid is *not* associative. But one can check that the above indeed

is a coherent 2-groupoid where associativity is *weakly* preserved in a *coherent* fashion, as described in [41].

Namely there are degenerate bigons for which $\mathrm{rank}(d\Sigma) \leq 1$, whose vertical composition with any other bigon has the effect of applying a thin homotopy to that bigon's source or target edges. Therefore associativity of horizontal composition of bigons holds up to vertical composition with such degenerate bigons and hence up to natural isomorphism.

11.3.2 p-Holonomy p-Functors

Our definition of 2-connection arises from categorifying the following definition of a 1-connection in a principal 1-bundle, which is equivalent to any of the other familiar definitions.

Definition 11.14 *A* **(1-)connection** *in a locally trivializable principal G-(1-)bundle $E \to B$ is the collection of the following data:*

- *A good covering $\mathcal{U} = \bigsqcup_{i \in I} U_i$ of B,*
- *for each $i \in I$ a smooth functor*

$$
\begin{array}{ccccc}
\mathrm{hol}_i : & \mathcal{P}_1(U_i) & \to & G & \\
& x \xrightarrow{\gamma} y & \mapsto & \bullet \xrightarrow{W_i[\gamma]} \bullet & ,
\end{array}
$$

  called the **local holonomy functor***, from the groupoid $\mathcal{P}_1(U_i)$ of paths in U_i (def. 11.10), to the structure group G (regarded as a category with a single object),*

- *for each $i, j \in I$ a natural isomorphism*

$$
\mathrm{hol}_i|_{U_{ij}} \xrightarrow{g_{ij}} \mathrm{hol}_j|_{U_{ij}} \,,
$$

  i.e. a commuting diagram

$$
\begin{tikzcd}
\bullet \arrow[r, "g_{ij}(x)"] \arrow[d, "W_i[\gamma]"'] & \bullet \arrow[d, "W_j[\gamma]"] \\
\bullet \arrow[r, "g_{ij}(y)"'] & \bullet
\end{tikzcd}
$$

  (for $x \xrightarrow{\gamma} y$ a path in U_{ij}) called the **transition natural isomorphisms** *between the local holonomy functors* hol_i *and* hol_j *restricted to U_{ij},*

- *and for each $i, j, k \in I$ a commuting diagram*

$$
\begin{tikzcd}
 & \mathrm{hol}_j \arrow[dr, "g_{jk}"] & \\
\mathrm{hol}_i \arrow[ur, "g_{ij}"] \arrow[rr, "g_{ik}"'] & & \mathrm{hol}_k
\end{tikzcd}
$$

equating the transition isomorphisms g_{ik} and $g_{ij} \circ g_{jk}$ restricted to U_{ijk} .

From this one obtains the familiar result that

1. The local holonomy functors hol_i are specified by 1-forms

$$A_i \in \Omega^1(U_i, \mathrm{Lie}(G)) \; .$$

2. The transition isomorphisms g_{ij} are specified by the transition functions

$$g_{ij} \colon U_{ij} \to G \, ,$$

satisfying the equation

$$A_i = g_{ij} A_j g_{ij}^{-1} + g_{ij} \mathbf{d} g_{ij}^{-1} \, , \qquad \text{on } U_{ij} \, .$$

3. The identity between these natural isomorphisms on triple overlaps U_{ijk} is equivalent to the equation

$$g_{ik} = g_{ij} g_{jk} \, , \qquad \text{on } U_{ijk} \, .$$

What we are after is a categorification of this situation. This leads to the following definition.

Definition 11.15 *A* **2-connection** *in a locally trivializable principal $\mathcal{G}$-2-bundle $E \to M$ which admits a* **2-holonomy** *is the collection of the following data:*

- *A good covering $\mathcal{U} = \bigsqcup_{i \in I} U_i$ of M,*
- *for each $i \in I$ a smooth 2-functor*

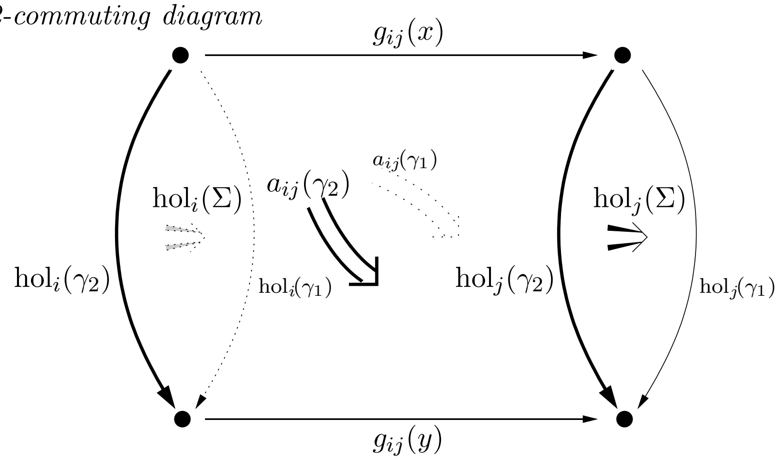
$$
\begin{array}{cccc}
\mathrm{hol}_i : & \mathcal{P}_2(U_i) & \to & G_2 \\
 & \begin{tikzcd} x \arrow[r, bend left=50, "\gamma"{name=U}] \arrow[r, bend right=50, "\gamma'"'{name=D}] \arrow[from=U, to=D, Rightarrow, "\Sigma"] & y \end{tikzcd} & \mapsto & \begin{tikzcd} \bullet \arrow[r, bend left=50, "\mathrm{hol}_i(\gamma)"{name=U}] \arrow[r, bend right=50, "\mathrm{hol}_i(\gamma')"'{name=D}] \arrow[from=U, to=D, Rightarrow, "\mathrm{hol}_i(\Sigma)"] & \bullet \end{tikzcd}
\end{array} , \tag{11.12}
$$

called the **local holonomy 2-functor***, from the 2-groupoid $\mathcal{P}_2(U_i)$ of 2-paths in U_i (def. 11.13, p. 256) to the structure 2-group $\mathcal{G}$ (regarded as a 2-category with a single object),*

- *for each $i, j \in I$ a pseudo-natural isomorphism*

$$\mathrm{hol}_i|_{U_{ij}} \xrightarrow{g_{ij}} \mathrm{hol}_j|_{U_{ij}} \,,$$

i.e. a 2-commuting diagram



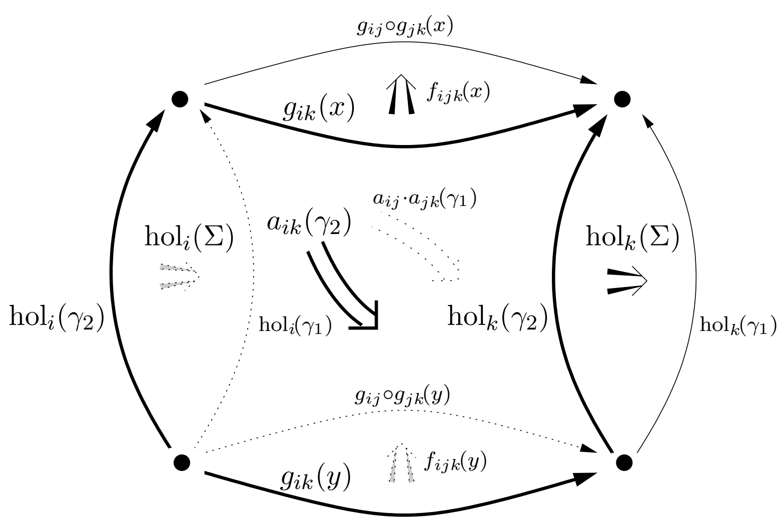
(11.13)

called the **transition pseudo-natural isomorphisms** *between the local holonomy 2-functors* hol_i *and* hol_j *restricted to* U_{ij}*,*

- *for each $i, j, k \in I$ a modification of pseudonatural transformations*

$$\begin{tikzcd}
 & \mathrm{hol}_j \arrow[dr, "g_{jk}"] & \\
\mathrm{hol}_i \arrow[ur, "g_{ij}"] \arrow[rr, "g_{ik}"'] \arrow[rr, phantom, shift left=4, "\Uparrow f_{ijk}"] & & \mathrm{hol}_k
\end{tikzcd}$$

i.e. a 2-commuting diagram



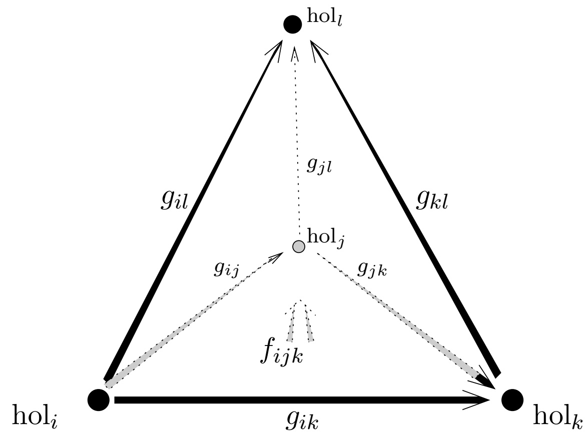
(11.14)

between the transition pseudonatural isomorphisms g_{ik} and $g_{ij} \circ g_{jk}$ restricted to U_{ijk},

- *for each $i, j, k, l \in I$ an identity*

$$
\begin{tikzcd}
\mathrm{hol}_j \arrow[r, "g_{jk}"] & \mathrm{hol}_k \arrow[d, "g_{kl}"] \\
\mathrm{hol}_i \arrow[u, "g_{ij}"] \arrow[ur, "g_{ik}"] \arrow[r, "g_{il}"'] \arrow[u, phantom, shift left=4ex, "\Uparrow f_{ijk}"] \arrow[r, phantom, shift left=4ex, "\Uparrow f_{ikl}"] & \mathrm{hol}_l
\end{tikzcd}
=
\begin{tikzcd}
\mathrm{hol}_j \arrow[r, "g_{jk}"] \arrow[dr, "g_{jl}"] \arrow[r, phantom, shift right=4ex, "\Uparrow f_{jkl}"] & \mathrm{hol}_k \arrow[d, "g_{kl}"] \\
\mathrm{hol}_i \arrow[u, "g_{ij}"] \arrow[r, "g_{il}"'] \arrow[r, phantom, shift left=4ex, "\Uparrow f_{ijl}"] & \mathrm{hol}_l
\end{tikzcd}
\tag{11.15}
$$

between these modifications, expressing the 2-commutativity of tetrahedra of the form



In analogy to the situation for 1-connections in 1-bundles, one would like to have an equivalent expression of this definition of a 2-connection in a 2-bundle in terms of differential forms. This is what we will work out in what follows. The result is this:

Proposition 11.2 *A 2-connection in a 2-bundle as defined in def. (11.15) is expressible in terms of differential forms as follows:*

1. *The local holonomy 2-funtor* hol_i *is specified by two differential forms*

$$
\begin{aligned}
A_i &\in \Omega^1(U_i, \mathfrak{g}) \\
B_i &\in \Omega^2(U_i, \mathfrak{h})
\end{aligned}
$$

satisfying

$$
F_{A_i} + dt(B_i) = 0\,, \tag{11.16}
$$

where F_{A_i} is the curvature 2-form of A_i.

The simple idea behind the proof for this is sketched in §11.4.1 (p.262). The full proof is the content of §11.5.3 (p.277).

2. *The transition pseudo-isomorphism* $\mathrm{hol}_i \xrightarrow{g_{ij}} \mathrm{hol}_j$ *is specified by*

$$
\begin{aligned}
g_{ij} &\in \Omega^0(U_{ij}, G) \\
a_{ij} &\in \Omega^1(U_{ij}, \mathfrak{h})
\end{aligned}
$$

   satisfying the equations

$$
\begin{aligned}
A_i &= g_{ij} A_j g_{ij}^{-1} + g_{ij} \mathbf{d} g_{ij}^{-1} \\
B_i &= \alpha(g_{ij})(B_i) + k_{ij}
\end{aligned}
\tag{11.17}
$$

   on U_{ij}*, where*

$$
k_{ij} = \mathbf{d} a_{ij} + a_{ij} \wedge a_{ij} + d\alpha(A_i) \wedge a_{ij} \,.
$$

   The proof for this is given in §11.4.2 (p.264). Again, the idea is quite simple, but the proof has to make use of some facts only developed in §11.5 (p.268).

3. *The modification* $g_{ik} \xrightarrow{f_{ijk}} g_{ij} \circ g_{jk}$ *is specified by*

$$
f_{ijk} \in \Omega^0(U_{ijk}, H)
$$

   satisfying the equation

$$
a_{ij} + g_{ij}(a_{jk}) - f_{ijk}\, a_{ik}\, f_{ijk}^{-1} - f_{ijk}\, \mathbf{d} h_{ijk}^{-1} - f_{ijk}\, d\alpha(A_i)\, (f_{ijk}^{-1}) = 0 \tag{11.18}
$$

   on U_{ijk}.

   This is proven in §11.4.3 (p.266).

4. *The equation between four of these modifications on quadruple overlaps is the already familiar tetrahedron law*

$$
\alpha(g_{ij})(f_{jkl})\; f_{ijl} = f_{ijk}\, f_{ikl} \,.
$$

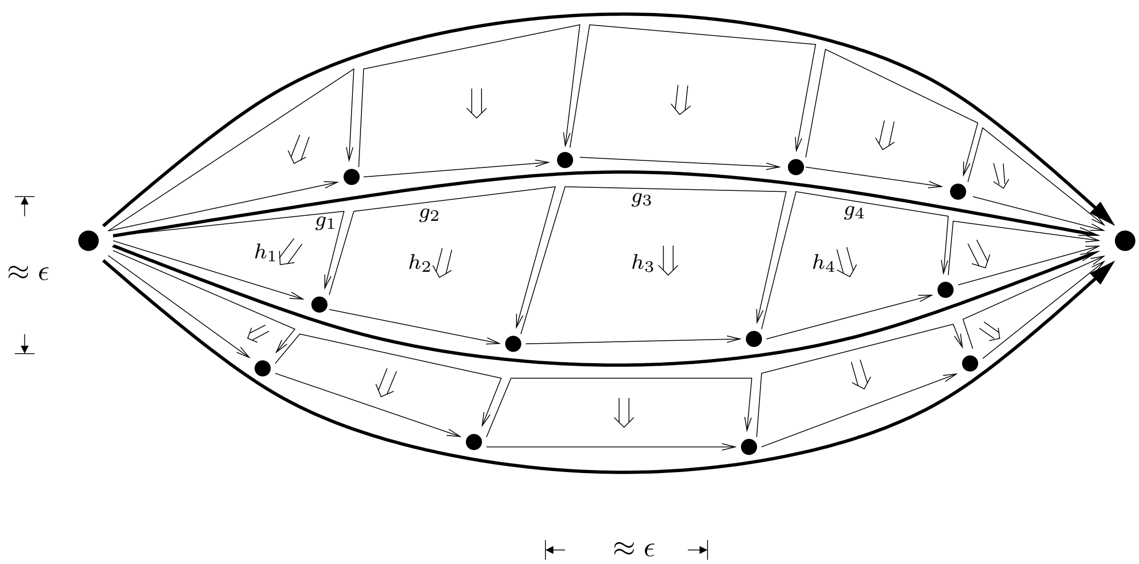
   This was discussed before in §11.2 (p.244).

11.4 2-Holonomy in Terms of Local p-Forms

In this subsection the proof of the central proposition 11.2 (p. 260) is sketched in a way that is supposed to clearly point out the underlying mechanisms in a concise way. Several technicalities that these proofs rely on are then discussed in detail in §11.5 (p.268).

11.4.1 Definition on Single Overlaps

Consider any bigon Σ in a patch U_i, i.e. a 2-morphism in $\mathcal{P}_2(U_i)$ (def. 11.13), and consider a local 2-holonomy functor $\mathrm{hol}_i \colon \mathcal{P}_2(U_i) \to \mathcal{G}$ (def. 11.15). Since hol_i is a functor, the 2-group 2-morphism which it associates to Σ can be computed by dividing Σ into many small sub-bigons, evaluating hol_i on each of these and composing the result in $\mathcal{G}$. This is illustrated in the following sketchy figure.



Here the j-th 2-morphism is supposed to be given by

$$\mathrm{hol}(\Sigma_j) = (g_j, h_j) \in \mathcal{G}$$

with $g \in G$ and $h \in H$. By the rules of 2-group multiplication (Prop. 9.1) the total horizontal product

$$(g^{\mathrm{tot}}, h^{\mathrm{tot}}) \equiv (g_1, h_1) \cdot (g_2, h_2) \cdot (g_3, h_3) \cdots$$

of all these 2-morphisms is given by

$$\begin{aligned} g^{\mathrm{tot}} &= g_1\, g_2\, g_3 \cdots g_N \\ h^{\mathrm{tot}} &= h_1\, \alpha(g_1)(h_2)\ \alpha(g_1 g_2)(h_3) \cdots \alpha(g_1 g_2 g_3 \cdots g_{N-1})(h_N)\ . \end{aligned}$$

The products of the g_j can be addressed as a *path holonomy* along the upper edges, which, for reasons to become clear shortly, we shall write as

$$g_1\, g_2 \cdots g_j \equiv (W_{j+1})^{-1}\,.$$

Now suppose the group elements come from algebra elements $A_j \in \mathfrak{g}$ and $B_j \in \mathfrak{h}$ as

$$\begin{aligned} g_j &\equiv \exp(\epsilon A_j) \\ h_j &\equiv \exp\big(\epsilon^2 B_j\big) \end{aligned} \tag{11.19}$$

where

$$\epsilon \equiv 1/N \,,$$

then

$$h^{\mathrm{tot}} = 1 + \epsilon^2 \sum_{j=1}^{N} \alpha\Big(W_j^{-1}\Big)(B_j) + \mathcal{O}\big(\epsilon^4\big) \ .$$

Using the notation

$$\begin{aligned} W_j &\equiv W(1 - j\epsilon, 1) \\ B_j &\equiv B(1 - \epsilon j) \end{aligned}$$

we have

$$h^{\mathrm{tot}} = 1 + \epsilon \int_0^1 d\sigma\ \alpha\big(W^{-1}(\sigma,1)\big)(B(\sigma)) + \mathcal{O}\big(\epsilon^3\big) \ .$$

Finally, imagine that the $\mathcal{G}$-labels h_k^{tot} of many such thin horizontal rows of 'surface elements' are composed *vertically*. Each of them comes from algebra elements

$$B_k(\sigma) \equiv B(\sigma, k\epsilon)$$

and holonomies

$$W_k(\sigma,1) \equiv W_{k\epsilon}(\sigma,1)$$

as

$$h_k^{\mathrm{tot}} \equiv 1 + \epsilon \int_0^1 d\sigma\ \alpha\big(W_{k\epsilon}^{-1}(\sigma,1)\big)(B(\sigma,k\epsilon)) + \mathcal{O}\big(\epsilon^3\big) \ .$$

In the limit of vanishing ϵ their total vertical product is

$$\lim_{\epsilon=1/N\to 0} h_0^{\mathrm{tot}} h_\epsilon^{\mathrm{tot}} h_{2\epsilon}^{\mathrm{tot}} \cdots h_1^{\mathrm{tot}} = \mathrm{P}\exp\left(\int_0^1 d\tau\ \mathcal{A}(\tau)\right)$$

for

$$\mathcal{A}(\tau) = \int_0^1 d\sigma\ \alpha\big(W_\tau^{-1}(\sigma,1)\big)(B(\sigma,\tau)) \ , \tag{11.20}$$

where P denotes path ordering with respect to τ.

Thinking of each of these vertical rows of surface elements as paths (in the limit $\epsilon \to 0$), this shows roughly how the computation of total 2-group elements from vertical and

horizontal products of many ‘small’ 2-group elements can be reformulated as the holonomy of a connection on path space of the form (11.20). This way the local 2-holonomy functor hol_i comes from a 1-form $A_i \in \Omega^1(U_i, \mathfrak{g})$ and a 2-form $B_i \in \Omega^2(U_i, \mathfrak{h})$ that arise as the continuum limit of the construction in (11.19). This is made precise in §11.5.3 (p.277).

There it is discussed that given a bigon $\gamma \xrightarrow{\Sigma} \tilde{\gamma}$ the 2-group morphism

$$\mathrm{hol}_i(\Sigma) = (W_i[\gamma_1] \in G,\ \mathcal{W}_i^{-1}[\Sigma] \in H) \tag{11.21}$$

is obtained from the holonomy $W_i[\gamma]$ of A_i along γ and the inverse of the path space holonomy $\mathcal{W}_i^{-1}[\Sigma]$ of $\mathcal{A}_{(A_i,B_i)}$ along a path in path space that maps to Σ.

But not every pair (A, B) corresponds to a local holonomy-functor. As first noticed in [92] there is a consistency condition which can be understood as follows:

Let $g_j \xrightarrow{h_j} g'_j$ be the jth 2-group 2-morphism in the above figure. The nature of 2-groups (prop. 9.1) requires that

$$t(h_j) = g'_j g_j^{-1}\,.$$

But, in the above sense, the left hand side is given by $\exp\Big(\epsilon^2 dt(B)_j\Big)$, while the right hand side is $\approx \exp\big(-\epsilon F_{A_j}\big)$, where F_{A_j} denotes the curvature 2-form of A evaluated on a 2-vector tangent to Σ_j. Hence we get the condition

$$dt(B) + F_A = 0\,.$$

This is the content of prop. 11.13 (p. 278). See also prop. 11.10 (p. 275).

11.4.2 Transition Law on Double Overlaps

Proposition 11.3. *The 2-commutativity of the diagram (11.13) (p. 259) is equivalent to the equations (11.17)*

$$\begin{aligned} A_i &= g_{ij} A_j g_{ij}^{-1} + g_{ij}\mathbf{d}g_{ij}^{-1} - dt(a_{ij}) \\ B_i &= \alpha(g_{ij})(B_i) + k_{ij}\,. \end{aligned}$$

Proof.

The 2-commutativity of the diagram is equivalent to the equality of the 2-morphism on its left face with the composition of the 2-morphisms on the front, back and right faces:

$$\begin{tikzcd} \bullet \arrow[r, bend left=50, "\mathrm{hol}_i(\gamma)"{name=U}] \arrow[r, bend right=50, "\mathrm{hol}_i(\tilde{\gamma})"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\mathrm{hol}_i(\Sigma)"] & \bullet \end{tikzcd} \quad = \quad \begin{tikzcd} \bullet \arrow[rrr, bend left=60, "\mathrm{hol}_i(\gamma)"] \arrow[rrr, bend right=60, "\mathrm{hol}_i(\tilde{\gamma})"'] \arrow[r, "g_{ij}(x)"] & \bullet \arrow[r, bend left=50, "\mathrm{hol}_j(\gamma)"{name=U2}] \arrow[r, bend right=50, "\mathrm{hol}_j(\tilde{\gamma})"'{name=D2}] \arrow[Rightarrow, from=U2, to=D2, "\mathrm{hol}_j(\Sigma)"] \arrow[r, phantom, shift left=9, "\Downarrow a_{ij}(\gamma)"] \arrow[r, phantom, shift right=9, "\Downarrow \bar{a}_{ij}(\tilde{\gamma})"] & \bullet \arrow[r, "g_{ij}^{-1}(y)"] & \bullet \end{tikzcd} \tag{11.22}$$

Recall from (11.21) that $\mathrm{hol}_i(\Sigma)$ has source $\mathrm{hol}_i(\gamma) = W_i[\gamma]$. So we write

$$a_{ij}(\gamma) \equiv (W_i[\gamma] \in G,\ E(a_{ij})\,[\gamma] \in H)\ ,$$

where

$$a_{ij} \in \Omega^1(U_{ij}, \mathfrak{h})$$

is a 1-form (which we find convenient to denote by the same symbol as the 2-morphism $a_{ij}(\gamma)$ that it is associated with) and where E is a function whose nature is to be determined by the source/target matching condition. This says that

$$t(E(a_{ij}[\gamma]))\,W_i[\gamma] \,=\, g_{ij}(x)\,W_j[\gamma]g_{ij}^{-1}(y)\ . \tag{11.23}$$

Expressions like this are handled by prop. 11.7 (p. 272). In order to apply it conveniently we take the inverse on both sides to get

$$W_i[\gamma^{-1}]t(E(a_{ij})\,[\gamma])^{-1} \,=\, g_{ij}W_j[\gamma^{-1}]g_{ij}^{-1} \tag{11.24}$$

(using $W[\gamma^{-1}] = W^{-1}[\gamma]$). Then the proposition tells us that $t(E(a_{ij})\,[\gamma])^{-1}$ is of the form

$$t(E(a_{ij})\,[\gamma])^{-1} \,=\, \lim_{\epsilon=1/N\to 0}\left(1+\epsilon\oint_{A_i}(\alpha)\right)\left(1+\epsilon\oint_{A_i+\epsilon\alpha}(\alpha)\right)\cdots\left(1+\epsilon\oint_{A_i+(1-\epsilon)a_{ij}^1}(\alpha)\right)_{|\gamma^{-1}}\ , \tag{11.25}$$

where the right hand side is evaluated at γ^{-1}, and where $\alpha \in \Omega^1(U_{ij}, \mathfrak{g})$ is given by

$$\alpha \equiv g_{ij}^1(\mathbf{d} + A_j)(g_{ij}^1)^{-1} - A_i\,.$$

The 1-form α must take values in the image of dt, and it is the corresponding pre-image which we denote by a_{ij}, so that $dt(a_{ij}) = \alpha$:

$$dt(a_{ij}) \,=\, g_{ij}^1(d + A_j)(g_{ij}^1)^{-1} - A_i\,. \tag{11.26}$$

This is the first of the two equations to be derived.

It follows that $E(a_{ij})\,[\gamma]$ itself is given by

$$(E(a_{ij})\,[\gamma])^{-1} \,=\, \lim_{\epsilon=1/N\to 0}\left(1+\epsilon\oint_{A_i}(a_{ij})\right)\left(1+\epsilon\oint_{A_i+\epsilon dt(a_{ij})}(a_{ij})\right)\cdots\left(1+\epsilon\oint_{A_i+(1-\epsilon)dt(a_{ij})}(a_{ij})\right)_{|\gamma^{-1}}\,.$$

Now that we have determined the 2-morphism $E(a_{ij})\,[\gamma]$, we can evaluate the diagrams in equation (11.22). Recalling again equation (11.21), one sees that the equality of the 2-morphism on the left hand with that on the right means that

$$\mathcal{W}_i^{-1}(\Sigma) = (E(a_{ij})\,[\tilde{\gamma}])^{-1}\mathcal{W}_j^{-1}(\Sigma)\,E(a_{ij})\,[\gamma]\,.$$

This is nothing but a gauge transformation of path space holonomy. Using Prop. 11.11 it implies the second of two equations to be proven. $\square$

Note that in the case that the 2-group in question is the abelian one given by the crossed module $(G = 1, H = U(1), \alpha = \text{trivial}, t = \text{trivial})$ the formula for $E(a_{ij})\,[\gamma]$ reduces simply to the line holonomy of a_{ij} along γ:

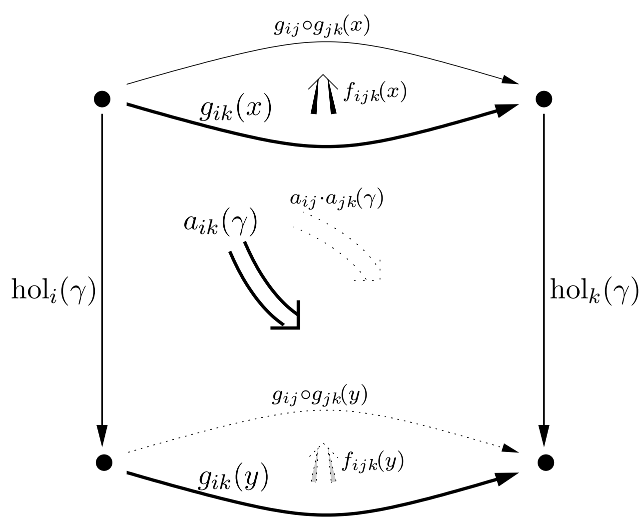
$$E(a_{ij})\,[\gamma] = \exp\left(\int_\gamma a_{ij}\right) . \tag{11.27}$$

11.4.3 Transition Law on Triple Overlaps

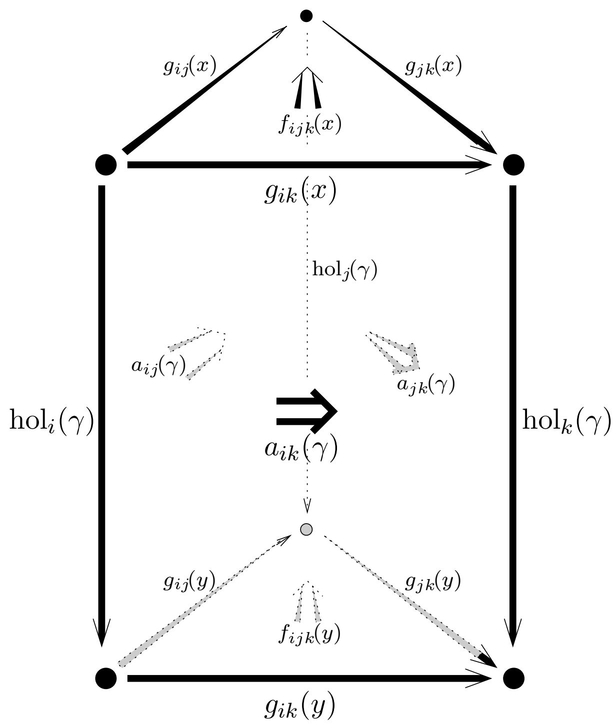
Proposition 11.4 *The 2-commutativity of the diagram (11.14) (p. 259) is equivalent to the equation (11.18) (p. 261)*

$$a_{ij} + g_{ij}(a_{jk}) - f_{ijk}\,a_{ik}\,f_{ijk}^{-1} - f_{ijk}\,\mathbf{d}f_{ijk}^{-1} - f_{ijk}\,d\alpha(A_i)\,(f_{ijk}^{-1}) = 0\,.$$

Proof. Since our target category $\mathcal{G}$ is a strict 2-group, so that (when regarded as a 2-category with a single object) all of its 1- and 2-morphisms are invertible, the diagram (11.14) expressing the modifications on U_{ijk} can be simplified. Using the transition diagram (11.13) we can equate the composition of the 2-morphisms $\mathrm{hol}_i(\Sigma)$ and $\mathrm{hol}_k(\Sigma)$ as well as the 2-morphisms $a_{ik}[\tilde{\gamma}]$ on the front of this diagram with the single 2-morphism $a_{ik}[\gamma]$ and hence get rid of the dependency on $\tilde{\gamma}$ and Σ:

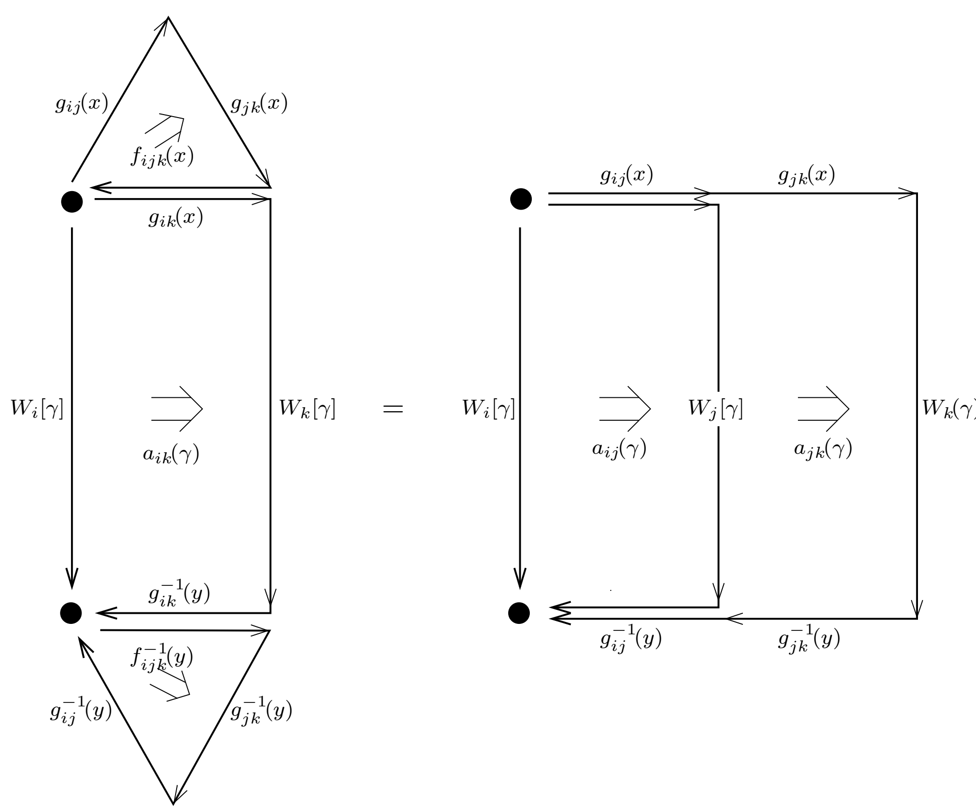


In order to emphasize the structure of this diagram it is useful to make the triangular shape of the top and bottom explicit:



(11.28)

The 2-commutativity of this diagram is equivalent to the following equality between the 2-morphism obtained from its top, bottom and front face and the 2-morphism obtained from the two faces on the back:



In terms of group elements this means that

$$f_{ijk}(x)\;\; E(a_{ik})\,[\gamma]\;\alpha(W_i[\gamma])\Big(f_{ijk}^{-1}(y)\Big) = \alpha(g_{ij}(x))(E(a_{jk})\,[\gamma])\;\; E(a_{ij})\,[\gamma]\,.$$

Now expand around the point x to get the differential version of this statemment:

$$\begin{aligned} W_i[\gamma] &\approx 1 + \epsilon A_i\big(\gamma'\big) \\ E(a_{ij})\,[\gamma] &\approx 1 + \epsilon a_{ij}\big(\gamma'\big) \\ f_{ijk}^{-1}(y) &\approx f_{ijk}^{-1}(x) + \epsilon(\mathbf{d}(f_{ijk}^{2})^{-1})\big(\gamma'\big)(x)\;. \end{aligned}$$

(Here $\gamma = \frac{d}{d\sigma}\gamma(0)$ is the tangent vector to γ at $x = \gamma(0)$.) Substituting this into the above equation and collecting terms of first order in ϵ yields the promised equation. □

11.5 Path Space

As we have seen, the space of all paths in a manifold or more general smooth spaces constitutes a smooth space it itself. In particular, we study the notion of holonomy for curves in path space. A curve in path space over U maps to a (possibly degenerate) surface in U and hence its path space holonomy gives rise to a notion of surface holonomy in U.

In this section we first discuss basic concepts of differential geometry on path spaces and then apply them to define path space holonomy. Using that, a 2-functor hol_i from the 2-groupoid of bigons in U_i (to be defined below) to the structure 2-group is defined and shown to be consistent.

Throughout the following, various p-forms taking values in Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ are used, where $\mathfrak{g}$ and $\mathfrak{h}$ are part of a differential crossed module $\mathcal{C}$ (def. 9.14).

Elements of a basis of $\mathfrak{g}$ will be denoted by T_a with $a \in (1, \ldots, \dim(\mathfrak{g}))$ and those of a basis of $\mathfrak{h}$ by S_a with $a \in (1, \ldots, \dim(\mathfrak{h}))$. Arbitrary elements will be expanded as $A = A^a T_a$.

Given a $\mathfrak{g}$-valued 1-form A its **gauge covariant exterior derivative** is

$$\begin{aligned} \mathbf{d}_A\omega &\equiv [\mathbf{d} + A, \omega] \\ &\equiv \mathbf{d}\omega + A^a \wedge d\alpha(T_a)(\omega) \end{aligned}$$

and its **curvature** is

$$\begin{aligned} F_A &\equiv (\mathbf{d} + A)^2 \\ &\equiv \mathbf{d}A + \frac{1}{2}A^a \wedge A^b\,[T_a, T_b]\;. \end{aligned}$$

By a **$\mathcal{C}$-valued (1,2)-form** on a manifold U we shall mean a pair (A, B) with

$$\begin{aligned} A &\in \Omega^1(U, \mathfrak{g}) \\ B &\in \Omega^2(U, \mathfrak{h})\;. \end{aligned} \tag{11.29}$$

Differential calculus on spaces of *parametrized* paths can be handled rather easily. We start by establishing some basic facts on parametrized paths and then define the *groupoid of paths* by considering thin homotopy equivalence classes of parametrized paths.

Definition 11.16. *Given a manifold U, the* **based parametrized path space** $P_s^t(U)$ *over U with source $s \in U$ and target $t \in U$ is the space of smooth maps*

$$\begin{aligned} X : [0,1] &\to U \\ \sigma &\mapsto X(\sigma) \end{aligned} \tag{11.30}$$

which are constant in a neighborhood of $\sigma = 0$ and in a neighborhood of $\sigma = 1$. When source and target coincide

$$\Omega_x(U) \equiv P_x^x(U)$$

is the **based loop space** *over U based at x.*

The constancy condition at the boundary is known as the property of having **sitting instant**, compare for instance [190]. It serves in def. 11.10 to ensure that the composition of two smooth parametrized paths is again a smooth parametrized path.

In the study of differential forms on parametrized path space the following notions play an important role (*cf.* [158], section 2):

Definition 11.17.

1. *Given any path space $P_s^t(U)$ (def. 11.16), the 1-parameter family of maps*

$$\begin{aligned} e_\sigma : P_s^t(U) &\to U \qquad (\sigma \in (0,1)) \\ \gamma &\mapsto \gamma(\sigma) \end{aligned}$$

   maps each path to its position in U at parameter value σ.

2. *Given any differential p-form $\omega \in \Omega^p(U)$ the pullback to $P_s^t(U)$ by e_σ shall be denoted simply by*

$$\omega(\sigma) \equiv e_\sigma^*(\omega) \ .$$

3. *The contraction of $\omega(\sigma)$ with the vector*

$$\gamma' \equiv \frac{d}{d\sigma}\gamma$$

   is denoted by $\iota_{\gamma'}\omega(\sigma)$.

A special class of differential forms on path space play a major role:

Definition 11.18. *Given a familiy $\{\omega_i\}_{i=1}^N$ of differential forms on a manifold U with degree*

$$\deg(\omega_i) \equiv p_i + 1$$

one gets a differential form (cf. (def. 11.17))

$$\Omega_{\{\omega_i\},(\alpha,\beta)}(\gamma) \equiv \oint\limits_{X|_\alpha^\beta} (\omega_1, \dots, \omega_n) \equiv \int\limits_{\alpha<\sigma_i<\sigma_{i+1}<\beta} \iota_{\gamma'}\omega_1\big(\sigma^1\big) \wedge \cdots \wedge \iota_{\gamma'}\omega_N\big(\sigma^N\big)$$

of degree

$$\deg\big(\Omega_{\{\omega_i\}}\big) = \sum_{i=1}^{N} p_i \,,$$

on any based parametrized path space $P_s^t(U)$ *(def. 11.16).*

For $\alpha = 0$ *and* $\beta = 1$ *we write*

$$\Omega_{\{\omega_i\}} \equiv \Omega_{\{\omega_i\},(0,1)} \,.$$

These path space forms are known as **multi integrals** *or* **iterated integrals** *or* **Chen forms** *(cf. [158, 53]).*

It turns out that the exterior derivative on path space maps Chen forms (def. 11.18) to Chen forms in a nice way:

Proposition 11.5. *The action of the path space exterior derivative on Chen forms (def. 11.18) is*

$$\mathbf{d} \oint (\omega_1, \cdots, \omega_n) = (\tilde{\mathbf{d}} + \tilde{M}) \oint (\omega_1, \cdots, \omega_n) \,, \tag{11.31}$$

where

$$\begin{aligned}
\tilde{\mathbf{d}} \oint (\omega_1, \cdots, \omega_n) &\equiv -\sum_k (-1)^{\sum\limits_{i<k} p_i} \oint (\omega_1, \cdots, \mathbf{d}\omega_k, \cdots, \omega_n) \\
\tilde{M} \oint (\omega_1, \cdots, \omega_n) &\equiv -\sum_k (-1)^{\sum\limits_{i<k} p_i} \oint (\omega_1, \cdots, \omega_{k-1} \wedge \omega_k, \cdots, \omega_n) \,,
\end{aligned}$$

satisfying

$$\begin{aligned}
\tilde{\mathbf{d}}^2 &= 0 \\
\tilde{M}^2 &= 0 \\
\left\{ \tilde{\mathbf{d}}^2, \tilde{M}^2 \right\} &= 0 \,.
\end{aligned} \tag{11.32}$$

(*cf.* [158, 53])

11.5.1 The Standard Connection 1-Form on Path Space

There are many 1-forms on path space that one could consider as local connection 1-forms in order to define a local holonomy on path space. Here we restrict attention to a special class, to be called the *standard connection 1-forms* (def. 11.21), because, as is shown in §11.5.3 (p.277), these turn out to be the ones which compute local 2-group holonomy. (This same 'standard connection 1-form' can however also be motivated from other points of view, as done in [160, 30].)

11.5.1.1 Holonomy and parallel transport. In order to set up some notation and conventions and for later references, the following gives a list of well-known definitions and facts that are crucial for the further developments:

Definition 11.19. *Given a path space $P_s^t(U)$ (def. 11.16) and a $\mathcal{C}$-valued (1,2)-form (A,B) (11.29) on U, the following objects are of interest:*

1. *The* **line holonomy** *of A along a given path γ is denoted by*

$$W_A[\gamma](\sigma^1,\sigma^2) \equiv \mathrm{P}\exp\left(\int_{\gamma|_{\sigma^1}^{\sigma^2}} A\right)$$
$$\equiv \sum_{n=0}^{\infty} \oint_{\gamma|_{\sigma^1}^{\sigma^2}} (A^{a_1},\dots,A^{a_n}) T_{a_1}\cdots T_{a_n}\,. \tag{11.33}$$

2. *The* **parallel transport** *of elements in $T\in\mathfrak{g}$ and $S\in\mathfrak{h}$ is written*

$$\begin{aligned}
T^{W_A[\gamma]}(\sigma) &\equiv W_A^{-1}[\gamma|_\sigma^1]T(\sigma)\,W_A[\gamma](\sigma,1)\\
&= \sum_{n=0}^{\infty}\oint_{\gamma|_\sigma^1}(-A^{a_1},\cdots,-A^{a_n})\ [T_{a_n},\cdots[T_{a_1},T(\sigma)]\cdots]\ ,\\
S^{W_A[\gamma]}(\sigma) &\equiv \alpha\big(W_A[\gamma|_\sigma^1]\big)(S(\sigma))\\
&\equiv \sum_{n=0}^{\infty}\oint_{\gamma|_\sigma^1}(-A^{a_1},\cdots,-A^{a_n})\ d\alpha(T_{a_n})\circ\cdots\circ d\alpha(T_{a_1})(S(\sigma))\ .
\end{aligned} \tag{11.34}$$

For convenience the dependency $[\gamma]$ on the path γ will often be omitted.

Proposition 11.6. *Parallel transport (def. 11.19) has the following properties:*

1. *Let $\sigma_1\le\sigma_2\le\sigma_3$ then*

$$W_A[\gamma](\sigma_1,\sigma_2)\circ W_A[\gamma](\sigma_2,\sigma_3) = W_A[\gamma](\sigma_1,\sigma_3)\ .$$

2. *Conjugation of elements in $\mathfrak{g}$ with parallel tranport of elements in $\mathfrak{h}$ yields*

$$W_A(\sigma,1)\big(d\alpha(T)(\sigma)\big(W_A^{-1}(\sigma,1)(S)\big)\big) = d\alpha\big(T^{W_A}(\sigma)\big)(S)\ . \tag{11.35}$$

3. *Given a G-valued 0-form $g\in\Omega^0(U,G)$ and a path $\gamma\in P_x^y(U)$ we have*

$$g(x)\,W_A[\gamma](g(y))^{-1} = W_{(gAg^{-1}+g^{-1}\mathbf{d}g)}[\gamma]\,. \tag{11.36}$$

4. *Given a G-valued 0-form $g \in \Omega^0(U, G)$ and a based loop $\gamma \in P_x^x(U)$ we have*

$$\alpha(\phi(x))(W_A[\gamma](\sigma, 1)(S(\sigma))) = W_{A'}[\gamma](\sigma, 1)(\alpha(\phi(\gamma(\sigma)))(S(\sigma))) \tag{11.37}$$

with

$$A' \equiv \phi A \phi^{-1} + \phi(d\phi^{-1}) .$$

Integrals over p-forms pulled back to a path and parallel tranported to some base point play an important role for path space holonomy. Following [53, 30] we introduce special notation to take care of that automatically:

Definition 11.20. *A natural addition to the notation (11.18) for iterated integrals in the presence of a $\mathfrak{g}$-valued 1-form A is the abbreviation*

$$\oint_A (\omega_1, \dots \omega_N) \equiv \oint \left(\omega_1^{W_A}, \dots \omega_N^{W_A}\right) ,$$

where $(\cdot)^{W_A}$ is defined in def 11.19. When Lie algebra indices are displayed on the left they are defined to pertain to the parallel tranported object:

$$\oint_A (\dots, \omega^a, \dots) \equiv \oint (\dots, (\omega^{W_A})^a, \dots) . \tag{11.38}$$

Using this notation first of all the following fact can be conveniently stated, which plays a central role in the analysis of the transition law for the 2-holonomy in §11.4.2 (p.264):

Proposition 11.7. *The difference in line holonomy (def. 11.19) along a given loop with respect to two different 1-forms A and A' can be expressed as*

$$(W_A[\gamma])^{-1} W_{A'}[\gamma] = \lim_{\epsilon = 1/N \to 0} \left(1 + \epsilon \oint_A (\alpha)\right) \left(1 + \epsilon \oint_{A + \epsilon(\alpha)} (\alpha)\right) \cdots \left(1 + \epsilon \oint_{A' - \epsilon(\alpha)} (\alpha)\right)_\gamma ,$$

with $\alpha \equiv A' - A$.

Proof.
First note that from def. 11.19 it follows that

$$\oint_A (\alpha) = \int_0^1 d\sigma (W_A[\gamma](\sigma, 1))^{-1} \iota_{\gamma'} \alpha(\sigma)\, W_A[\gamma](\sigma, 1) .$$

This implies that

$$W_A[\gamma] \left(1 + \epsilon \oint_A (\alpha)\right)_\gamma = W_{A + \epsilon(\alpha)}[\gamma] + \mathcal{O}(\epsilon^2) .$$

The proposition follows by iterating this. $\square$

11.5.1.2 Exterior derivative and curvature for Chen forms. The exterior derivative on path space maps Chen forms to Chen forms (Prop. 11.5). Since we shall be interested in Chen forms involving parallel transport (def. 11.20), it is important to know also the particular action of the exterior derivative on these:

Proposition 11.8. *The action of the path space exterior derivative on $\oint_A(\omega)$ is*

$$\mathbf{d}\oint_A(\omega) = -\oint_A(\mathbf{d}_A\omega) - (-1)^{\deg(\omega)}\oint_A(d\alpha(T_a)(\omega)\,,F_A^a)\,. \tag{11.39}$$

(Recall the convention (11.38)).
Proof.
This is a straightforward, though somewhat tedious, computation using prop 11.5. □

We have restricted attention here to just a single insertion, i.e. $\oint_A(\omega)$ instead of $\oint_A(\omega_1,\ldots,\omega_n)$, because this is the form that the *standard connection 1-form* has:

Definition 11.21. *Given a $\mathcal{C}$-valued $(1,2)$-form (11.29) the path space 1-form*

$$\Omega^1\big(P_s^t(U)\,,\mathfrak{h}\big) \ni \mathcal{A}_{(A,B)} \equiv \oint_A(B)\,.$$

is here called the **standard local connection 1-form on path space**.

(*cf.* [160, 161, 30])
Given a connection, one wants to know its curvature:

Corollary 11.1 *The curvature of the standard path space 1-form $\mathcal{A}_{(A,B)}$ (def. 11.21) is*

$$\mathcal{F}_{\mathcal{A}} = -\oint_A(\mathbf{d}_A B) - \oint_A(d\alpha(T_a)(B)\,,(F_A + dt(B))^a)\,. \tag{11.40}$$

Proof. Use Prop. 11.8. □

Definition 11.22. *Given a standard path space connection 1-form $\mathcal{A}_{(A,B)}$ (def. 11.21) coming from a $\mathfrak{g}$-valued 1-form A and an $\mathfrak{h}$-valued 2-form B*

- *the 3-form*

$$H \equiv \mathbf{d}_A B \tag{11.41}$$

  is called the **curvature 3-form**,

- *the 2-form*

$$\tilde{F} \equiv F_A + dt(B) \tag{11.42}$$

  is called the **fake curvature 2-form**.

The term 'fake curvature' has been introduced in [49]. The notation $\tilde{F}$ follows [92]. The curvature 3-form was used in [21].

Using this notation the local path space curvature reads

$$\mathcal{F}_{\mathcal{A}} = -\oint_A (H) - \oint_A \Big(d\alpha(T_a)(B)\,,\tilde{F}^a \Big)\,. \tag{11.43}$$

11.5.2 Path Space Line Holonomy and Gauge Transformations

With the usual tools of differential geometry available for path space the holonomy on path space is defined as usual:

Definition 11.23. *Given a path space 1-form $\mathcal{A}$ and a curve Σ in path space the* **path space line holonomy** *of $\mathcal{A}$ along Σ is*

$$\mathcal{W}_{\mathcal{A}}(\Sigma) \equiv \mathrm{P}\exp\Big(\int_\Sigma \mathcal{A}\Big)\,.$$

Note that by definition P here indicates path ordering with objects at higher parameter value to the *right* of those with lower parameter value, just as in the definition of ordinary line holonomy in (def. 11.19).

Path space line holonomy has a richer set of gauge transformations than holonomy on base space. In fact, ordinary gauge transformations on base space correspond to *constant* ('global') gauge transformations on path space in the following sense:

Proposition 11.9. *Given a path space line holonomy (def. 11.23) coming from a standard path space connection 1-form (def. 11.21) $\mathcal{A}_{(A,B)}$ in a based loop space $P^x_x(U)$ as well as a G-valued 0-form $\phi \in \Omega^0(U,G)$ we have*

$$\alpha(\phi(x))\Big(\mathcal{W}_{\mathcal{A}_{(A,B)}}(\Sigma)\Big) = \mathcal{W}_{\mathcal{A}_{(A',B')}}(\Sigma)$$

with

$$\begin{aligned} A' &= \phi A \phi^{-1} + \phi(d\phi^{-1}) \\ B' &= \alpha(\phi)(B)\,. \end{aligned}$$

Proof. Write out the path space holonomy in infinitesimal steps and apply (11.37) on each of them. □

The usual notion of gauge transformation is obtained by conjugation:

Definition 11.24. *Given the path space holonomy $\mathcal{W}_{\mathcal{A}_{(A,B)}}(\Sigma|^{\gamma_1}_{\gamma_0})$ (def. 11.23) of a standard local path space connection 1-form $\mathcal{A}_{(A,B)}$ (def. 11.21) along a curve Σ in $P^t_s(U)$ with endpaths γ_0 and γ_1, an* **infinitesimal path space holonomy gauge transformation** *is a 1-parameter familiy maps*

$$\begin{aligned} \mathcal{W}_{\mathcal{A}_{(A,B)}}(\Sigma|^{\gamma_1}_{\gamma_0}) \mapsto & \Big(1 - \epsilon \oint_A (a)\Big)_{\gamma_0} \mathcal{W}_{\mathcal{A}_{(A,B)}}(\Sigma|^{\gamma_1}_{\gamma_0}) \Big(1 + \epsilon \oint_A (a)\Big)_{\gamma_1} \\ & \equiv \mathrm{Ad}^{\gamma_1}_{\gamma_0}\Big(1 - \epsilon \oint_A (a)\Big)\Big(\mathcal{W}_{\mathcal{A}_{(A,B)}}(\Sigma|^{\gamma_1}_{\gamma_0})\Big)\,, \end{aligned}$$

for $\epsilon \in \mathbb{R}$ and for a any 1-form

$$a \in \Omega^1(U, \mathfrak{h}) \,.$$

This yields a new sort of gauge transformation in terms of the target space (1,2) form (A, B):

Proposition 11.10. *Infinitesimal path space holonomy gauge transformations (def. 11.24) for the holonomy of a standard path space connection 1-form $\mathcal{A}_{(A,B)}$ and arbitrary transformation parameter a yields to first order in the parameter ϵ the path space holonomy of a transformed standard path space connection 1-form $\mathcal{A}_{(A',B')}$ with*

$$\begin{aligned} A' &= A + dt(a) \\ B' &= B - \mathbf{d}_A a \end{aligned} \tag{11.44}$$

if and only if the fake curvature (def. 11.22) vanishes.

(This was originally considered in [30] for the special case $G = H$, $t = \mathrm{id}$, $\alpha = \mathrm{Ad}$.)
Proof.

As for any holonomy, the gauge transformation induces a transformation of the connection 1-form $\mathcal{A} \to \mathcal{A}'$ given by

$$\begin{aligned} \mathcal{A}' &= \left(1 - \epsilon \oint_A (a)\right)(\mathbf{d} + \mathcal{A})\left(1 + \epsilon \oint_A (a)\right) \\ &= \mathcal{A} + \epsilon\, \mathbf{d}_{\mathcal{A}} \oint_A (a) + \mathcal{O}(\epsilon^2) \,. \end{aligned} \tag{11.45}$$

Using (11.39) one finds (using the notation (11.38))

$$\mathcal{A} + \epsilon\, \mathbf{d}_{\mathcal{A}} \oint_A (a) = \oint_{A'} (B') + \epsilon \oint_A (d\alpha(T_a)(a)\,, (dt(B) + F)^a) + \mathcal{O}(\epsilon^2) \,.$$

Since a is by assumption arbitrary, the last line is equal to a standard connection 1-form to order ϵ if and only if $dt(B) + F = 0$. $\square$

The above infinitesimal gauge transformation is easily integrated to a finite gauge transformation:

Definition 11.25. *A* **finite path space holonomy gauge transformation** *is the integration of infinitesimal path space holonomy gauge transformations (def. 11.24), i.e. it is a map for any $a \in \Omega^1(U, \mathfrak{h})$ given by*

$$\mathcal{W}_{\mathcal{A}_{(A,B)}}(\Sigma|_{\gamma_0}^{\gamma_1}) \mapsto \lim_{\epsilon = 1/N \to 0} \underbrace{\mathrm{Ad}_{\gamma_0}^{\gamma_1}\left(1 - \epsilon \oint_{A + dt(a)} (a)\right) \cdots \mathrm{Ad}_{\gamma_0}^{\gamma_1}\left(1 - \epsilon \oint_A (a)\right)}_{N \text{ factors}} \left(\mathcal{W}_{\mathcal{A}_{(A,B)}}(\Sigma|_{\gamma_0}^{\gamma_1})\right) \,.$$

Proposition 11.11. *A finite path space holonomy gauge transformation (def. 11.25) of the holonomy of a standard path space connection 1-form $\mathcal{A}_{(A,B)}$ is equivalent to a transformation*

$$\mathcal{A}_{(A,B)} \mapsto \mathcal{A}_{(A',B')}$$

where

$$\begin{aligned} A &\mapsto A + dt(a) \\ B &\mapsto B - \underbrace{(d_A a + a \wedge a)}_{\equiv k_a} \end{aligned} \tag{11.46}$$

is the transformed $(1,2)$*-form* (A,B)*.*

Proof. This is a standard computation. □

In summary the above yields two different notions of gauge transformations on path space:

1. If the path space in question is a based loop space then according to Prop. 11.9 a gauge transformation on target space yields an ordinary gauge transformation of the $(1,2)$-form (A,B):

   $$\begin{aligned} A &\mapsto \phi A \phi^{-1} + \phi(d\phi^{-1}) \\ B &\mapsto \alpha(\phi)(B)\ . \end{aligned}$$

   We shall call this a **2-gauge transformation of the first kind**.

2. A gauge transformation in path space itself yields, according to prop. 11.11, a transformation

   $$\begin{aligned} A &\mapsto A + dt(a) \\ B &\mapsto B - (d_A a + a \wedge a)\,. \end{aligned}$$

   We shall call this a **2-gauge transformation of the second kind.**

Recall that according to Prop. 11.10 this works precisely when (A,B) defines a standard connection 1-form (def. 11.21) on path space for which the 'fake curvature' (def. 11.22) vanishes $\tilde{F} = dt(B) + F_A = 0$.

In the context of loop space these two transformations and the conditions on them were discussed for the special case $G = H$ and $t = \mathrm{id}$, $\alpha = \mathrm{Ad}$ in [30]. In the context of 2-groups and higher lattice gauge theory they were found in section 3.4 of [92]. They also appear in the transition laws for nonabelian gerbes [49, 50, 23], as is discussed in detail in §9.4 (p.194). The same transformation for the special case where all groups are abelian is well known from abelian gerbe theory [47] but also for instance from string theory (e.g. section 8.7 of [129]).

With holonomy on path space understood, it is now possible to use the fact that every curve in path space maps to a (possibly degenerate) surface in target space in order to get a notion of (local) surface holonomy. That is the content of the next subsection.

11.5.3 The local 2-Holonomy Functor

Definition 11.26. *Given a patch U and a 2-group $\mathcal{G}$ a* **local 2-holonomy** *is a strict 2-functor*

$$\mathrm{hol}\colon \mathcal{P}_2(U) \to \mathcal{G}$$

from the path 2-groupoid $\mathcal{P}_2(U)$ (def. 11.13) to the 2-group $\mathcal{G}$.

(The fact that this functor is strict means that it ignores the parametrization of the bigons' source and target edges. Eventually one may want to replace the structure 2-group here with the more general 'coherent' 2-group dicussed in [41], and the strict 2-functor with a more general sort of 2-functor.)

We want to construct a local 2-holonomy from a standard path space connection 1-form (def. 11.21). In order to do so we first construct a 'pre-2-holonomy' for any standard path space connection 1-form and then determine under which conditions this actually gives a true 2-holonomy. It turns out that the necessary and sufficient conditions for this is the vanishing of the fake curvature (def. 11.22).

Definition 11.27. *Given a standard path space connection 1-form (def. 11.21) and given any parametrized bigon (def. 11.11) $\Sigma : [0,1]^2 \to U$ with source edge $\gamma_1 \equiv \Sigma(\cdot, 0)$ and target edge $\gamma_2 \equiv \Sigma(\cdot, 1)$, the triple $(g_1, h, g_2) \in G \times H \times G$ with*

$$\begin{aligned} g_i &\equiv W_A(\gamma_i) \\ h &\equiv \mathcal{W}_{\mathcal{A}}^{-1}(\Sigma(1 - \cdot, \cdot)) \end{aligned} \qquad (11.47)$$

is called the **local pre-2-holonomy** *of Σ associated with $\mathcal{A}$.*

In order for a pre-2-holonomy to give rise to a true 2-holonomy two conditions have to be satisfied:

1. The triple (g_1, h, g_2) has to specify a 2-group element. By Prop. 9.1 this is the case precisely if $g_2 = t(h)\, g_1$ (9.2).
2. The pre-2-holonomy has to be invariant under thin homotopy in order to be well defined on bigons.

The solution of this is the content of Prop. 11.15 below. In order to get there the following considerations are necessary:

In order to analyze the first of the above two points consider the behaviour of the pre-2-holonomy under changes of the target edge.

Given a path space $P_s^t(U)$ and a $\mathfrak{g}$-valued 1-form with line holonomy holonomy $W_A[\gamma]$ on $\gamma \in P_s^t$ (def. 11.19) the **change in holonomy** of W_A as one changes γ is well known to be given by the following:

Proposition 11.12. *Let $\rho : \tau \mapsto \gamma(\tau)$ be the flow generated by the vector field D on P_s^t, then*

$$\frac{d}{d\tau} W_A^{-1}[\gamma(0)] W_A[\gamma(\tau)]\Big|_{\tau=0} = -\left(\oint_A (F_A)\right)(D) \,. \tag{11.48}$$

(Note that the right hand side denotes evaluation of the path space 1-form $\oint_A (F_A)$ on the path space vector field D.)

Proof. The proof is standard. The only subtlety is to take care of the various conventions for signs and orientations which give rise to the minus sign in (11.48). □

Proposition 11.13. *For the pre-2-holonomy (def. 11.27) of parametrized bigons Σ associated with the standard connection 1-form $\mathcal{A}_{(A,B)}$ to specify 2-group elements, i.e. for the triples (g_1, h, g_2) to satisfy $g_2 = t(h)\, g_1$, we must have*

$$dt(B) + F_A = 0\,.$$

Proof. According to def. 11.27 the condition $g_2 = t(h)\, g_1$ translates into

$$\begin{aligned} t(h) &= W_A(\gamma_2)\, W_A^{-1}(\gamma_1) \\ &= W_A^{-1}\big(\gamma_2^{-1}\big)\, W_A\big(\gamma_1^{-1}\big) \,. \end{aligned}$$

Now let there be a flow $\tau \mapsto \gamma_\tau$ on $P_s^t(U)$ generated by a vector field D and choose $\gamma_2^{-1} = \gamma_\tau$ and $\gamma_1^{-1} = \gamma_0$. Then according to Prop. 11.12 we have

$$\frac{d}{d\tau} W_A^{-1}\big(\gamma_2^{-1}\big)\, W_A\big(\gamma_1^{-1}\big) = +\left(\oint_A (F_A)\right)_{\gamma_0} (D) \,,$$

where the plus sign is due to the fact that D here points opposite to the D in Prop. 11.12.

Applying the same τ-derivative on the left hand side of (11.49) yields

$$-\left(\oint_A (dt(B))\right)(D) = \left(\oint_A (F_A)\right)(D) \,.$$

(Here the minus sign on the left hand side comes from the fact that we have identified $t(h)$ with the *inverse* path space holonomy $\mathcal{W}_{\mathcal{A}}^{-1}$. This is necessary because the ordinary path space holonomy is path-ordered to the right, while we need $t(h)$ to be path ordered to the left.)

This can be true for all D only if $-dt(B) = F_A$. □

This is nothing but the **nonabelian Stokes theorem**. (Compare for instance [191] and references given there.)

Next it needs to be shown that a pre-2-holonomy with $dt(B) + F_A = 0$ is invariant under thin-homotopy:

Proposition 11.14. *The standard path space connection 1-form $\mathcal{A}_{(A,B)}$ (def. 11.21) is inavriant under thin homotopy precisely if the path space 2-form*

$$\oint_A (d\alpha(T_a)(B)\,, (F_A + dt(B))^a) \tag{11.49}$$

vanishes on all pairs path space vector fields that generate thin homotopy flows.

Proof. For the special case $G = H$ and $t = \mathrm{id}$, $\alpha = \mathrm{Ad}$ this was proven by [192]. The full proof is a straightforward generalization of this special case:

Consider a path Σ in path space with tangent vector T and let D be any vector field on $P_s^t(U)$. By a standard result the path space holonomy $\mathcal{W}(\Sigma)$ is invariant under the flow generated by D iff the curvature of $\mathcal{A}$ vanishes on T and D, $\mathcal{F}(T,D) = 0$.

But from corollary 11.1 we know that $\mathcal{F} = -\oint(\mathbf{d}_A B) - \oint(d\alpha(T_a)(B)\,, (F_A + dt(B))^a)$. It is easy to see that $\oint(\mathbf{d}_A B)$ vanishes on all pairs of tangent vectors that generate thin homotopy transformations of Σ and that the remaining term vanishes on (T,D) for all D if it vanishes on all pairs of tangent vector that generate thin homotopy transformations. $\square$

Now we can finally prove the following:

Proposition 11.15. *The pre-2-holonomy (def. 11.27) induces a true local 2-holonomy (def. 11.26)*

$$\begin{array}{cccc} \mathrm{hol}_i : & \mathcal{P}_2(U_i) & \to & G_2 \\ & \begin{tikzcd} x \arrow[r, bend left=40, "\gamma"{name=U}] \arrow[r, bend right=40, "\gamma'"'{name=D}] & y \arrow[Rightarrow, from=U, to=D, "\Sigma"] \end{tikzcd} & \mapsto & \begin{tikzcd} \bullet \arrow[r, bend left=40, "W_i[\gamma]"{name=U}] \arrow[r, bend right=40, "W_i[\gamma']"'{name=D}] & \bullet \arrow[Rightarrow, from=U, to=D, "\mathcal{W}_i^{-1}(\Sigma)"] \end{tikzcd} \end{array}$$

precisely if the fake curvature (def. 11.22) vanishes.

Proof.

We have already shown that for $dt(B) + F_A = 0$ the pre-2-holonomy indeed maps into a 2-group (Prop. 11.13) and that its values are well defined on bigons (Prop. 11.14). What remains to be shown is functoriality, i.e. that the pre-2-holonomy respects the composition of bigons and 2-group elements.

First of all it is immediate that composition of paths is respected, due to the properties of ordinary holonomy. Vertical composition of 2-holonomy (being composition of ordinary holonomy in path space) is completely analogous. The fact that pre-2-holonomy involves the *inverse* path space holonomy takes care of the nature of the vertical product in the

2-group, which reverses the order of factors: In the diagram

$$
\begin{tikzcd}
\mathcal{G} & \bullet \arrow[rr, bend left=60, "W_A[\gamma_1]", ""{name=U1, below}] \arrow[rr, "W_A[\gamma_2]" description, ""{name=M1}] \arrow[rr, bend right=60, "W_A[\gamma_3]"', ""{name=D1, above}] \arrow[Rightarrow, from=U1, to=M1, "\mathcal{W}_{\mathcal{A}}^{-1}[\Sigma_1]"] \arrow[Rightarrow, from=M1, to=D1, "\mathcal{W}_{\mathcal{A}}^{-1}[\Sigma_2]"] & & \bullet & = & \bullet \arrow[rr, bend left, "W_A[\gamma_1]"] \arrow[rr, bend right, "W_A[\gamma_3]"'] \arrow[rr, phantom, "\Downarrow \mathcal{W}_{\mathcal{A}}^{-1}[\Sigma_1\circ\Sigma_2]"] & & \bullet \\
\Uparrow \mathrm{hol} & & \Uparrow & & & & \Uparrow & \\
\mathcal{P}_2(U) & x \arrow[rr, bend left=60, "\gamma_1", ""{name=U2, below}] \arrow[rr, "\gamma_2" description, ""{name=M2}] \arrow[rr, bend right=60, "\gamma_3"', ""{name=D2, above}] \arrow[Rightarrow, from=U2, to=M2, "{[\Sigma_1]}"] \arrow[Rightarrow, from=M2, to=D2, "{[\Sigma_2]}"] & & y & = & x \arrow[rr, bend left, "\gamma_1"] \arrow[rr, bend right, "\gamma_3"'] \arrow[rr, phantom, "\Downarrow {[\Sigma_1\circ\Sigma_2]}"] & & y
\end{tikzcd}
$$

the top right bigon must be labeled (according to the properties of 2-groups described in Prop. 9.1) by

$$
\begin{aligned}
(W_A[\gamma_1], \mathcal{W}_{\mathcal{A}}^{-1}[\Sigma_1]) \circ (W_A[\gamma_2], \mathcal{W}_{\mathcal{A}}^{-1}[\Sigma_2]) &= (W_A[\gamma_1], \mathcal{W}_{\mathcal{A}}^{-1}[\Sigma_2]\mathcal{W}_{\mathcal{A}}^{-1}[\Sigma_1]) \\
&= (W_A[\gamma_1], \mathcal{W}_{\mathcal{A}}^{-1}[\Sigma_1 \circ \Sigma_2])\,,
\end{aligned}
$$

which indeed is the label associated by the hol-functor in the right column of the diagram.

So far we have suppressed in these formulas the reversal (11.47) in the first coordinate of Σ, since it plays no role for the above. This reversal however is essential in order for the hol-functor to respect horizontal composition.

In order to see this it is sufficient to consider *whiskering*, i.e. horizontal composition with identity 2-morphisms.

When whiskering from the left we have

$$
\begin{tikzcd}
\mathcal{G} & \bullet \arrow[r, "W_A[\gamma_1]"] & \bullet \arrow[r, bend left, "W_A[\gamma_2]"] \arrow[r, bend right, "W_A[\gamma_2']"'] \arrow[r, phantom, "\Downarrow \mathcal{W}_{\mathcal{A}}^{-1}[\Sigma]"] & \bullet & = & \bullet \arrow[r, bend left, "W_A[\gamma_1\circ\gamma_2]"] \arrow[r, bend right, "W_A[\gamma_1\circ\gamma_2']"'] \arrow[r, phantom, "\Downarrow \alpha(W_A[\gamma_1])(\mathcal{W}_{\mathcal{A}}^{-1}[\Sigma])"] & \bullet \\
\Uparrow \mathrm{hol} & & \Uparrow & & & \Uparrow & \\
\mathcal{P}_2(U) & x \arrow[r, "\gamma_1"] & y \arrow[r, bend left, "\gamma_2"] \arrow[r, bend right, "\gamma_2'"'] \arrow[r, phantom, "\Downarrow {[\Sigma]}"] & z & = & x \arrow[r, bend left, "\gamma_1\circ\gamma_2"] \arrow[r, bend right, "\gamma_1\circ\gamma_2'"'] \arrow[r, phantom, "\Downarrow {[\Sigma]}"] & z
\end{tikzcd}
$$

Evaluating the line holonomy in path space for this situation involves taking the path ordered exponential of (*cf.* (11.34))

$$
\int\limits_{(\gamma_1\circ\gamma_2)^{-1}} d\sigma\ \alpha\big(W_A^{-1}[(\gamma_1 \circ \gamma_2)^{-1}|_\sigma^1]\big)(B(\sigma))
$$

evaluated on the tangent vector to the whiskered Σ. Since this vanishes on γ_1 and using the reparameterization invariance of W_A the above equals

$$\cdots = \alpha(W_A[\gamma_1])\left(\int\limits_{\gamma_2^{-1}} d\sigma\ \alpha\big(W_A^{-1}[\gamma_2^{-1}|_\sigma^1]\big)(B(\sigma))\right) .$$

Hence the above diagram does commute. In this computation the path reversal is essential, which of course is related to our convention that parallel transport be to the point with parameter $\sigma = 1$. A simple plausibility argument for this was given at the beginning of §11.5.1 (p.270).

Finally, whiskering to the right is trivial, since we can simply use reparametrization invariance to obtain

$$\int\limits_{(\gamma_1\circ\gamma_2)^{-1}} d\sigma\ \alpha\big(W_A[(\gamma_1\circ\gamma_2)^{-1}|_\sigma^1]\big)(B(\sigma)) = \int\limits_{\gamma_1^{-1}} d\sigma\ \alpha\big(W_A[\gamma_1^{-1}|_{\sigma^1}]\big)(B(\sigma)) ,$$

because for right whiskers the integrand vanishes on γ_2.

Since general horizontal composition is obtained by first whiskering and then composing vertically, this also proves that the hol-functor respects general horizontal composition.

In summary, this shows that a pre-2-holonomy with vanishing fake curvature (def. 11.22) $dt(B) + F_A = 0$ defines a 2-functor hol: $\mathcal{P}_2(U) \to \mathcal{G}$ and hence a local strict 2-holonomy. $\square$

11.6 2-Curvature

Since curvature is the first order term in the holonomy around a small loop, the 2-transition Prop. 11.3 of 2-holonomy immediately implies a transition law for the path space curvature 2-form $\mathcal{F}_{\mathcal{A}} = -\oint_A(H)$ (11.43) and hence of the curvature 3-form $H = \mathbf{d}_A B$ (def. 11.22).

First of all one notes the following:

Proposition 11.16. *The curvature 3-form (def. 11.22) $H = \mathbf{d}_A B$ transforms* covariantly *under gauge transformations of the first kind (11.47). Moreover, it is* invariant *under gauge transformations of the second kind (11.47) if and only if the fake curvature vanishes.*

Proof.

The covariant transformation under gauge transformations of the first kind follows from simple standard reasoning. The invariance under infinitesimal transformations of the second kind with $A \to A + \epsilon dt(a)$ and $B \to B - \epsilon \mathbf{d}_A a$ follows from noting the invariance

under ‘infinitesimal’ shifts:

$$
\begin{aligned}
H = \mathbf{d}_A B \quad &\to \quad \mathbf{d}_{A+\epsilon dt(a)}(B - \epsilon \mathbf{d}_A a) \\
&= \quad \mathbf{d}_A B - \epsilon \left(\mathbf{d}_A \mathbf{d}_A a - d\alpha(dt(a))(B)\right) + \mathcal{O}\big(\epsilon^2\big) \\
&\overset{(9.6)}{=} \quad H - \epsilon \left(d\alpha(F_A)(a) + d\alpha(dt(B))(a)\right) + \mathcal{O}\big(\epsilon^2\big) \\
&\overset{(11.42)}{=} \quad H - \epsilon d\alpha\Big(\tilde{F}\Big)(B) + \mathcal{O}\big(\epsilon^2\big) \\
&\overset{\tilde{F}=0}{=} \quad H + \mathcal{O}\big(\epsilon^2\big)\,.
\end{aligned}
\tag{11.50}
$$

□

(*cf.* equation (3.43) of [92]).

Note that the invariance of H under transformations of the second kind does imply invariance of the path space curvature 2-form $\mathcal{F}_{\mathcal{A}}$. Naively, this transforms as

$$
\mathcal{F}_{\mathcal{A}} = -\oint_A (H) \to -\oint_{A+dt(a)} (H)\,.
$$

but since $\mathrm{im}(dt)$ acts trivially on $\ker(dt)$ (this is shown in prop. 11.17 right below) we have

$$
-\oint_{A+dt(a)} (H) = -\oint_A (H)\,.
$$

Proposition 11.17 *In a crossed module* $\mathrm{im}(t)$ *acts trivially on* $\ker(t)$. *Equivalently, in a differential crossed module* $\mathrm{im}(dt)$ *acts trivially on* $\ker(dt)$.

Proof. This is a consequence of the property

$$
\begin{aligned}
\alpha(t(h_1))(h_2) &= h_1 h_2 h_1^{-1} \\
d\alpha(dt(S_1))(S_2) &= [S_1, S_2]
\end{aligned}
\tag{11.51}
$$

of a crossed module, with $h_i \in H$ and $S_i \in \mathfrak{h}$:

Let $h \in H$ and $k \in \ker(t) \subset H$. Then

$$
\begin{aligned}
\alpha(t(h))(k) &= hkh^{-1} \\
&= kk^{-1}hkh^{-1} \\
&= k\,\underbrace{\alpha\big(t\big(k^{-1}\big)\big)(h)}_{=h}\,h^{-1} \\
&= k\,.
\end{aligned}
$$

Similarly for differential crossed modules with $S \in \mathfrak{h}$ and $S_0 \in \ker(dt) \subset \mathfrak{h}$:

$$
\begin{aligned}
d\alpha(dt(S))(S_0) &= [S, S_0] \\
&= -\,[S_0, S] \\
&= -\,\underbrace{d\alpha(dt(S_0))}_{=0}(S) \\
&= 0\,.
\end{aligned}
\tag{11.52}
$$

□

The transition law for $H_i \equiv \mathbf{d}_{A_i} B_i$ is now a simple corollary:

Corollary 11.2 *The local curvature 3-form $H_i = \mathbf{d}_{A_i} B_i$ of the local standard path space connection of a 2-bundle with 2-connection has the transition law*

$$H_i = \alpha\big(g^1_{ij}\big)(H_j)$$

on double intersections U_{ij}.

This is the transition law (9.29) of the curvature 3-form of a nonabelian gerbe for vanishing fake curvature and $d_{ij} = 0$.

One should note that also the fake curvature (def. 11.22) transforms covariantly and can therefore indeed consistently be chosen to vanish: The transition law for F_{A_i} following from (11.26) is

$$F_{A_i} = g_{ij} F_{A_j} g_{ij}^{-1} - dt(k_{ij})$$

and that of $dt(B)$ following from Prop. 11.3 (p. 264)

$$dt(B_i) = g_{ij}\, dt(B_j)\, g_{ij}^{-1} + dt(k_{ij}) \;,$$

so that

$$\tilde{F}_i = g_{ij} \tilde{F}_j g_{ij}^{-1}\,.$$

The **Bianchi-identity** on path space says that

$$\begin{aligned}
0 &= \mathbf{d} \oint_A (H) + \underbrace{\oint_A (B) \wedge \oint_A (H) - \oint_A (H) \wedge \oint_A (B)}_{= 0 \text{ by prop. 11.17}} \\
&= \mathbf{d} \oint_A (H) \\
&= - \oint_A (\mathbf{d}_A H) - \oint_A (d\alpha(T_a)(H)\,, F^a) \\
&= - \oint_A (\mathbf{d}_A H) + \oint_A (d\alpha(T_a)(H)\,, dt(B)^a) \\
&= \oint_A (\mathbf{d}_A H)\;.
\end{aligned}$$

Hence the vanishing of the fake curvature ensures that the 3-form field strength is covariantly closed:

$$\mathbf{d}_A H = 0\,.$$

Since $\mathbf{d}_A H = (\mathbf{d}_A)^2 B = F_A \wedge B$ this can be seen more explicitly also as follows:

Proposition 11.18 *The vanishing of the fake curvature implies that*

$$F_A \wedge B = 0\,,$$

which is shorthand for

$$(F_A^a \wedge B^b)\; d\alpha(T_a)(S_b) = 0\,.$$

Proof. Use $F_A = -dt(B)$ to get

$$\begin{aligned}
(F_A^a \wedge B^b)\; d\alpha(T_a)(S_b) &= -(B^a \wedge B^b)\; d\alpha(dt(S_a))(S_b) \\
&= -(B^a \wedge B^b)\; [S_a, S_b] \\
&= 0\,.
\end{aligned}$$

This vanishes because $B^a \wedge B^b = B^b \wedge B^a$ (since B is a 2-form) while $[S_a, S_b] = -[S_b, S_a]$.
$\square$

This again ensures that self-duality of the field strength, i.e.

$$H = \pm \star H$$

is sufficient to imply equations of motion of the form

$$\mathbf{d}_A \star H = 0\,.$$

In the abelian case this ensures that the 6-dimensional self-dual theory compactifies to an ordinary gauge theory (*cf.* §4.1.1 (p.64)). Vanishing fake curvature also ensures that this is gauge invariant.[12]

[12]These observations concerning the equation $\mathbf{d}_A H = 0$ arose in a discussion with Jens Fjelstad.

12. Global 2-Holonomy

In §11 (p.242) [31] 2-bundles with 2-connection admitting 2-holonomy have been described, and the transition laws for the 2-connection have been worked out. With these ingredients it is now possible to write down a definition of global surface holonomy obtained by gluing together local surface holonomies. This is discussed in the following.

The result is a diagrammatic explanation and generalization to nonabelian and, with slight modifications, weakened cases of the procedure that was apparently first stated by Alvarez [42] in the context of 2D topological field theories and later used by Gawedzki and Reis [45] in the context of the WZW model.

Note on Notation:

We will display many 2-commuting diagrams in the following. In order not to overburden the labeling of the 1- and 2-morphism we will frequently leave "re-whiskering" of 2-morphisms implicit. This means that whenever a displayed 2-morphism does not go between its genuine source and target 1-morphisms we have implicitly composed it horizontally with appropriate identity 2-morphisms so that it goes between source and target as indicated in the respective diagram. This should never lead to any ambiguities but instead make the diagrams more easily readable.

12.1 1-Bundles with 1-Connection

We start by considering the situation that we are interested in for the case of ordinary (1-)bundles with (1-)connection.

Recall the definition of a principal 1-bundle with 1-connection from def 11.14 (p. 257) (see also §3.3.1 (p.55)). This was given in terms of local holonomy functors which were related on double overlaps by natural transformations. One can easily see that this can equivalently be described by a *single* functor, called the **global (1-)holonomy (1-)functor** from what we call the "Čech -extended" path (1-)groupoid to the structure group.

12.1.1 The global 1-Holonomy 1-Functor

There is a groupoid, called the Čech -groupoid, whose morphisms are the "transitions" of any given point from one patch U_i of a good covering $\mathcal{U}$ to another patch U_j:

Definition 12.1 *The* **Čech (1-)groupoid** $C(\mathcal{U})$ *of a good covering* $\mathcal{U} \to M$ *has as objects all points in* $\mathcal{U}$

$$\mathrm{Ob}(C(U)) \equiv \{(x,i)|i \in I\,, x \in U_i\}$$

and its morphisms are formal arrows

$$\mathrm{Mor}(C(U)) \equiv \{(x,i) \to (x,j)|i,j \in I\,, x \in U_{ij}\}$$

such that there is at most one morphism between any two objects, i.e. such that every triangle

$$\begin{tikzcd} & (x,j) \arrow[dr] & \\ (x,i) \arrow[ur] \arrow[rr] & & (x,k) \end{tikzcd}$$

commutes.

We can combine this Čech -groupoid with the path groupoids $\mathcal{P}_1(U_i)$ of all the patches U_i (def. 11.10) into a single groupoid which we call the **Čech -extended path groupoid** $\mathcal{P}_1^C(\mathcal{U})$.

Definition 12.2 *The* **Čech -extended path (1-)groupoid** $\mathcal{P}_1^C(\mathcal{U})$ *of a good covering* $\mathcal{U} \to M$ *has objects*

$$\mathrm{Ob}\big(\mathcal{P}_1^C(\mathcal{U})\big) \equiv \{(x,i) | i \in I \,, x \in U_i\}$$

and its set of morphisms is defined to be the set of morphism generated by formal compositions of the elements in $\mathrm{Mor}(\mathcal{P}_1(U_i))$ $, \forall\, i \in I$ *and* $\mathrm{Mor}(C(\mathcal{U}))$*, divided out by all commuting diagrams of the form*

$$\begin{tikzcd} (x,i) \arrow[r] \arrow[d, "{[\gamma_i]}"'] & (x,j) \arrow[d, "{[\gamma_j]}"] \\ (y,i) \arrow[r] & (y,j) \end{tikzcd}$$

whenever $[\gamma_i]|_{U_{ij}} = [\gamma_j]|_{U_{ij}}$

Heuristically, this diagram expresses how a path in a double overlap of two patches can be regarded as a path in either of the two patches. More technically, we can regard this diagram as expressing a certain natural isomorphism between two functors that map a given pre-image path to $[\gamma_i]$ and to $[\gamma_j]$, respectively.

We can now define a **global holonomy functor** hol to be any functor hol from the Čech -extended path groupoid to the structure group G:

$$\mathrm{hol} : \mathcal{P}_1^C(U) \to G\,.$$

Given any such functor, denote its image on $[\gamma_i]$ by

$$\mathrm{hol}([\gamma_i]) \equiv \mathrm{hol}_i(\gamma)$$

and denote its image of $(x,i) \to (x,j)$ by

$$\mathrm{hol}((x,i) \to (x,j)) \equiv g_{ij}(x)\ .$$

Then the application of hol to the commuting diagram from (def. 12.1) yields the cocycle condition on the transition function

$$
\mathrm{hol}\left(
\begin{tikzcd}
 & (x,j) \arrow[dr] & \\
(x,i) \arrow[ur] \arrow[rr] & & (x,k)
\end{tikzcd}
\right)
=
\begin{tikzcd}
 & \bullet \arrow[dr, "g_{jk}"] & \\
\bullet \arrow[ur, "g_{ij}"] \arrow[rr, "g_{ik}"'] & & \bullet
\end{tikzcd}
\quad ,
$$

while application of hol to the commuting diagram from def. 12.1 yields

$$
\mathrm{hol}\left(
\begin{tikzcd}
(x,i) \arrow[r] \arrow[d, "{[\gamma_i]}"'] & (x,j) \arrow[d, "{[\gamma_j]}"] \\
(y,i) \arrow[r] & (y,j)
\end{tikzcd}
\right)
=
\begin{tikzcd}
\bullet \arrow[r, "g_{ij}(x)"] \arrow[d, "{\mathrm{hol}_i([\gamma])}"'] & \bullet \arrow[d, "{\mathrm{hol}_j([\gamma])}"] \\
\bullet \arrow[r, "g_{ij}(y)"'] & \bullet
\end{tikzcd}
,
$$

which is the transition law for the local holonomy functor hol_i.

So the global functor hol does capture all the information about the local holonomy functors hol_i together with their gluing conditions.

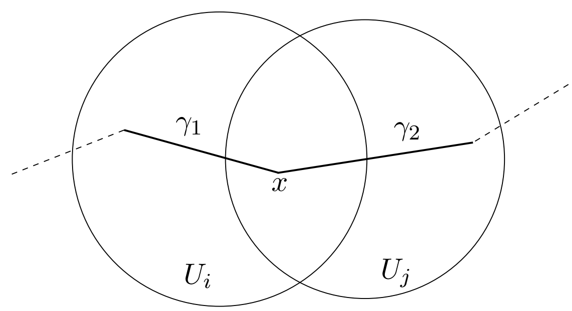
Suppose we want to compute some holonomy of a (class of a) path $[\gamma] \in \mathrm{Mor}(\mathcal{P}_1(M))$ in the base manifold M. This path will in general not sit inside a single patch U_i. We can however "lift" it to a morphism in the Čech -extended path groupoid $\mathcal{P}_1^C(\mathcal{U})$ by dividing it into N sub-paths $[\gamma_n]\,, n \in \{1, 2, \dots, N\}$ that all do sit in a single U_i,

$$[\gamma_n] \in \mathrm{Mor}(P(U_{i_n}))\ ,$$

and composing these with morphisms in $C(\mathcal{U})$:

$$(\gamma_0(0)\,, i_0) \xrightarrow{[\gamma_0]} (\gamma_0(1)\,, i_0) \to (\gamma_1(0)\,, i_1) \xrightarrow{[\gamma_1]} (\gamma_1(1)\,, i_1) \to \cdots \to (\gamma_N(0)\,, i_N) \xrightarrow{[\gamma_N]} (\gamma_N(1)\,, i_N)\,.$$

On this morphism we can apply hol and regard the result as the holonomy of the original path $[\gamma]$ (with respect to the given choice of trivialization at the endpoints of $[\gamma]$).



$$\Big\downarrow \mathrm{hol}$$

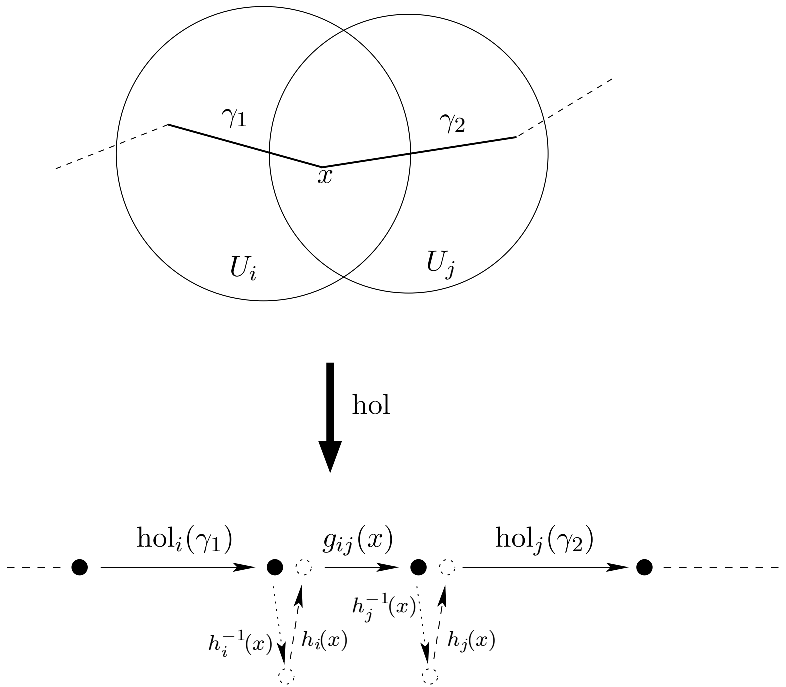
$$\cdots \bullet \xrightarrow{\mathrm{hol}_i(\gamma_1)} \bullet \xrightarrow{g_{ij}(x)} \bullet \xrightarrow{\mathrm{hol}_j(\gamma_2)} \bullet \cdots$$

The result reproduces the familiar law for how to compute global line holonomy. This is well defined since a gauge transformation amounts to

$$\bullet \xrightarrow{\tilde{g}_{ij}} \bullet \quad = \quad \bullet \xrightarrow{h_i} \bullet \xrightarrow{g_{ij}} \bullet \xrightarrow{h_j^{-1}} \bullet$$

$$\bullet \xrightarrow{\tilde{\mathrm{hol}}_i} \bullet \quad = \quad \bullet \xrightarrow{h_i} \bullet \xrightarrow{\mathrm{hol}_i} \bullet \xrightarrow{h_i^{-1}} \bullet$$

and since these two contributions cancel:



From the point of view of the global holonomy functor this fact becomes a tautology, becuase gauge transformations of the local trivialization of the bundle with connection are nothing but natural transformations of the global holonomy functor, as can be seen from the following naturality squares:

$$\begin{array}{c|c}
\mathcal{P}_1^C(\mathcal{U}) & G \\
\hline
\begin{tikzcd}
\bullet\ (x,i) \arrow[d, "\gamma_1"] \\
\bullet\ (y,i) \arrow[d] \\
\bullet\ (y,j) \arrow[d, "\gamma_2"] \\
\bullet\ (z,j)
\end{tikzcd}
&
\begin{tikzcd}
\bullet \arrow[r, "h_i(x)"] \arrow[d, "\mathrm{hol}_i(\gamma_1)"'] & \bullet \arrow[d, "\tilde{\mathrm{hol}}_i(\gamma_1)"] \\
\bullet \arrow[r, "h_i(y)"] \arrow[d, "g_{ij}(y)"'] & \bullet \arrow[d, "\tilde{g}_{ij}(y)"] \\
\bullet \arrow[r, "h_j(y)"'] \arrow[d, "\mathrm{hol}_j(\gamma_2)"'] & \bullet \arrow[d, "\tilde{\mathrm{hol}}_j(\gamma_2)"] \\
\bullet \arrow[r, "h_j(y)"'] & \bullet
\end{tikzcd}
\end{array} \tag{12.1}$$

Here

$$\mathrm{hol} \xrightarrow{h} \tilde{\mathrm{hol}}$$

is a natural transformation between global holonomy functors hol and $\tilde{\mathrm{hol}}$, given by

$$h((x,i)) \equiv h_i(x) \in G\,.$$

12.2 2-Bundles with 2-Connection

The above considerations for 1-connections with 1-holonomy in 1-bundles are straightforwardly generalized to 2-connection with 2-holonomy in 2-bundles.

12.2.1 The Čech -extended 2-Path 2-Groupoid

We are again interested in merging the 2-path 2-groupoids $\mathcal{P}_2(U_i)$ (def. 11.13) with the Čech 2-groupoid $C_2(\mathcal{U})$ of the covering $\mathcal{U} = \bigsqcup\limits_{i \in I} U_i$, which is the generalization of the Čech -1-groupoid from def. 12.1.

Definition 12.3 *Given a good covering $\mathcal{U} \to M$, the* **Čech 2-groupoid** $C_2(\mathcal{U})$ *is defined as follows: Its objects are all the elements*

$$\mathrm{Ob}(C_2(\mathcal{U})) = \{(x,i) | i \in I\,, x \in U_i\}$$

of the covering $\mathcal{U}$ and its 1-morphisms are those generated *from the 1-morphisms present in $C(\mathcal{U})$,*

$$\mathrm{Mor}_1(C_2(U)) = \langle \{(x,i) \to (x,j) | i \in I\,, x \in U_i\} \rangle \;,$$

but we no longer demand to have commuting triangle diagrams. In particular, for every object (x,i) there is now precisely one morphisms

$$(x,i) \to (x,i)$$

not equal to the identity morphism (which we denote by $\mathrm{Id}_{(x,i)}$).

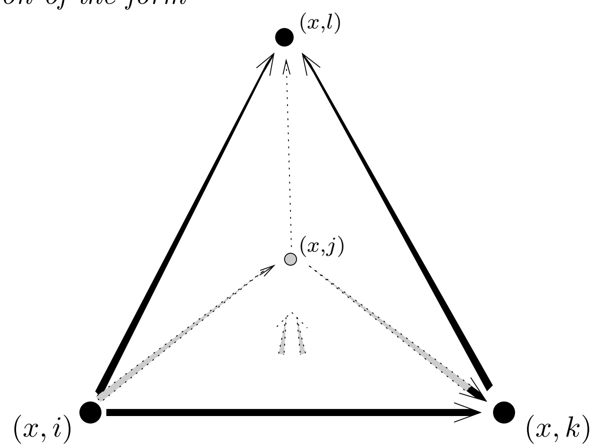
In addition, there is in $C_2(\mathcal{U})$ precisely one 2-morphism between any two 1-morphisms with coinciding endpoints:

$$\mathrm{Mor}_2(C_2(U)) = \left\{ \begin{array}{c} (x,i) \to (x,j_1) \to \cdots (x,j_n) \to (x,k) \\ \Downarrow \\ (x,i) \to (x,j'_1) \to \cdots (x,j'_m) \to (x,k) \end{array} \right\} .$$

Hence instead of 1-commuting triangles we have traingle 2-morphisms

$$\begin{tikzcd} & (x,j) \arrow[dr] \arrow[d, phantom, "\Uparrow"] & \\ (x,i) \arrow[ur] \arrow[rr] & & (x,k) \end{tikzcd}$$

and every tetrahedron of the form



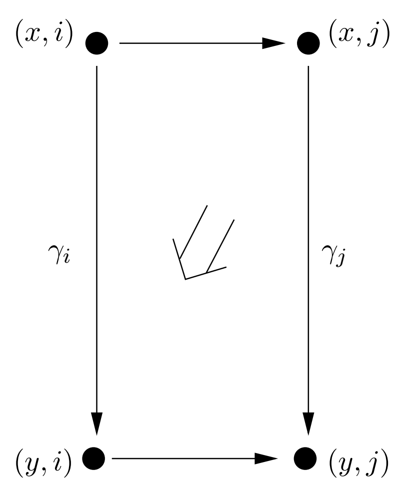
(12.2)

2-commutes.

As before, the 2-groupoids $C_2(\mathcal{U})$ and $\mathcal{P}_2(U_i)$, $i \in I$, can be merged to what we shall call the *Čech -extended 2-path 2-groupoid* $\mathcal{P}_2^C(\mathcal{U})$ of the covering $\mathcal{U} \to M$.

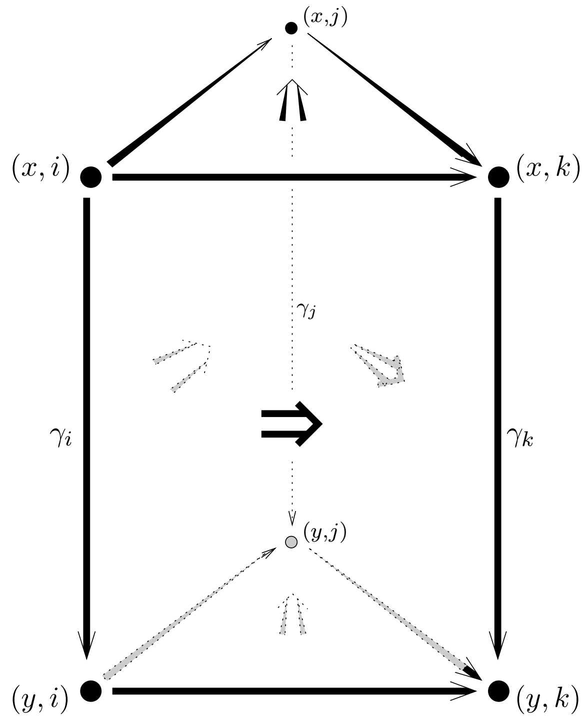
Definition 12.4 *The* **Čech -extended 2-path 2-groupoid** *of a covering* $\mathcal{U} \to M$ *is defined as follows:*

The set of objects is the same as before, the set of 1-morphisms is that generated by formally composing those of $\mathcal{P}_2(U_i)\,, i \in I$*, and* $C_2(\mathcal{U})$*. The set of 2-morphisms are the formal compositions generated by the 2-morphisms in* $\mathcal{P}_2(U_i)\,, i \in I$*, and* $C_2(\mathcal{U})$ *and in addition we throw in precisely one 2-morphism*



whenever $\gamma_i|_{U_{ij}} = \gamma_j|_{U_{ij}}$*. This expresses how a path in a double overlap of two patches can be regarded as a path in either of the two patches.*

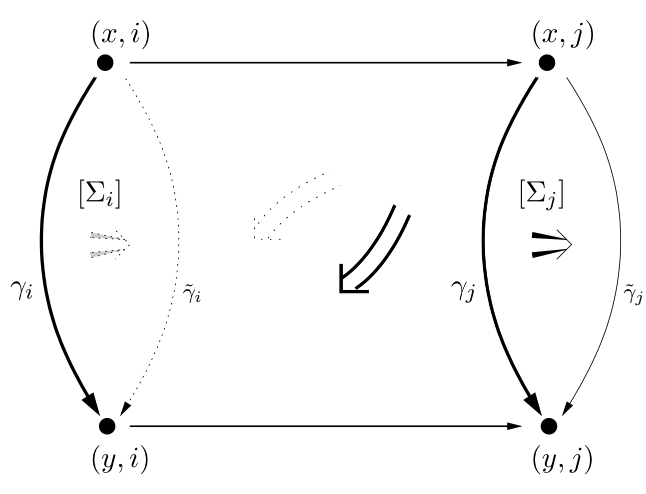
It follows that in $\mathcal{P}_2^C(\mathcal{U})$ we have 2-commuting diagrams of the following form



(12.3)

This is because the front side has the same boundary as the rest and since by the above there is precisely one 2-morphism for any boundary of this form.

But now there are also surfaces (bigons) sitting in double overlaps. For these we similarly need to postulate a 2-commuting ("tin can"-) diagram expressing how they can be realized as bigons in either of the two patches:



(12.4)

12.2.2 The global 2-Holonomy 2-Functor

We can now define a *global holonomy 2-functor* to be any 2-functor hol from the Čech

-extended 2-path 2-groupoid to the structure 2-group:

$$\mathrm{hol} : \mathcal{P}_2^C(\mathcal{U}) \to G_2\,.$$

As before, it is a matter of introducing notation to define the following quantities:

$$\mathrm{hol}\left(\begin{tikzcd}(x,i) \arrow[r, bend left] & (x,j)\end{tikzcd}\right) \equiv \begin{tikzcd}\bullet \arrow[r, bend left, "g_{ij}(x)"] & \bullet\end{tikzcd}$$

$$\mathrm{hol}\left(\begin{tikzcd}(x,i) \arrow[r, bend left, ""{name=U, below}] \arrow[r, bend right, "\mathrm{Id}"', ""{name=D, above}] \arrow[Rightarrow, from=U, to=D] & (x,i)\end{tikzcd}\right) \equiv \begin{tikzcd}\bullet \arrow[r, bend left, "g_{ii}(x)", ""{name=U, below}] \arrow[r, bend right, "1"', ""{name=D, above}] \arrow[Rightarrow, from=U, to=D, "k_i(x)"] & \bullet\end{tikzcd}$$

$$\mathrm{hol}\left(\begin{tikzcd} & (x,j) \arrow[dr] & \\ (x,i) \arrow[ur] \arrow[rr] \arrow[rr, phantom, bend left=20, "\Uparrow"] & & (x,k)\end{tikzcd}\right) \equiv \begin{tikzcd} & \bullet \arrow[dr, "g_{jk}(x)"] & \\ \bullet \arrow[ur, "g_{ij}(x)"] \arrow[rr, "g_{ik}(x)"'] \arrow[rr, phantom, bend left=20, "\Uparrow f_{ijk}(x)"] & & \bullet\end{tikzcd}$$

$$\mathrm{hol}\left(\begin{tikzcd}(x,i) \arrow[r] \arrow[d, "\gamma_i"'] \arrow[dr, phantom, "\swarrow"] & (x,j) \arrow[d, "\gamma_j"] \\ (y,i) \arrow[r] & (y,j)\end{tikzcd}\right) \equiv \begin{tikzcd}\bullet \arrow[r, "g_{ij}(x)"] \arrow[d, "\mathrm{hol}_i(\gamma)"'] \arrow[dr, phantom, "\swarrow a_{ij}(\gamma)"] & \bullet \arrow[d, "\mathrm{hol}_j(\gamma)"] \\ \bullet \arrow[r, "g_{ij}(y)"'] & \bullet\end{tikzcd}$$

$$\mathrm{hol}\left(\begin{tikzcd}(x,i) \arrow[r, bend left, "\gamma_1", ""{name=U, below}] \arrow[r, bend right, "\gamma_2"', ""{name=D, above}] \arrow[Rightarrow, from=U, to=D, "{[\Sigma]}"] & (y,i)\end{tikzcd}\right) \equiv \begin{tikzcd}\bullet \arrow[r, bend left, "\mathrm{hol}_i(\gamma_1)", ""{name=U, below}] \arrow[r, bend right, "\mathrm{hol}_i(\gamma_2)"', ""{name=D, above}] \arrow[Rightarrow, from=U, to=D, "\mathrm{hol}(\Sigma)"] & \bullet\end{tikzcd}$$

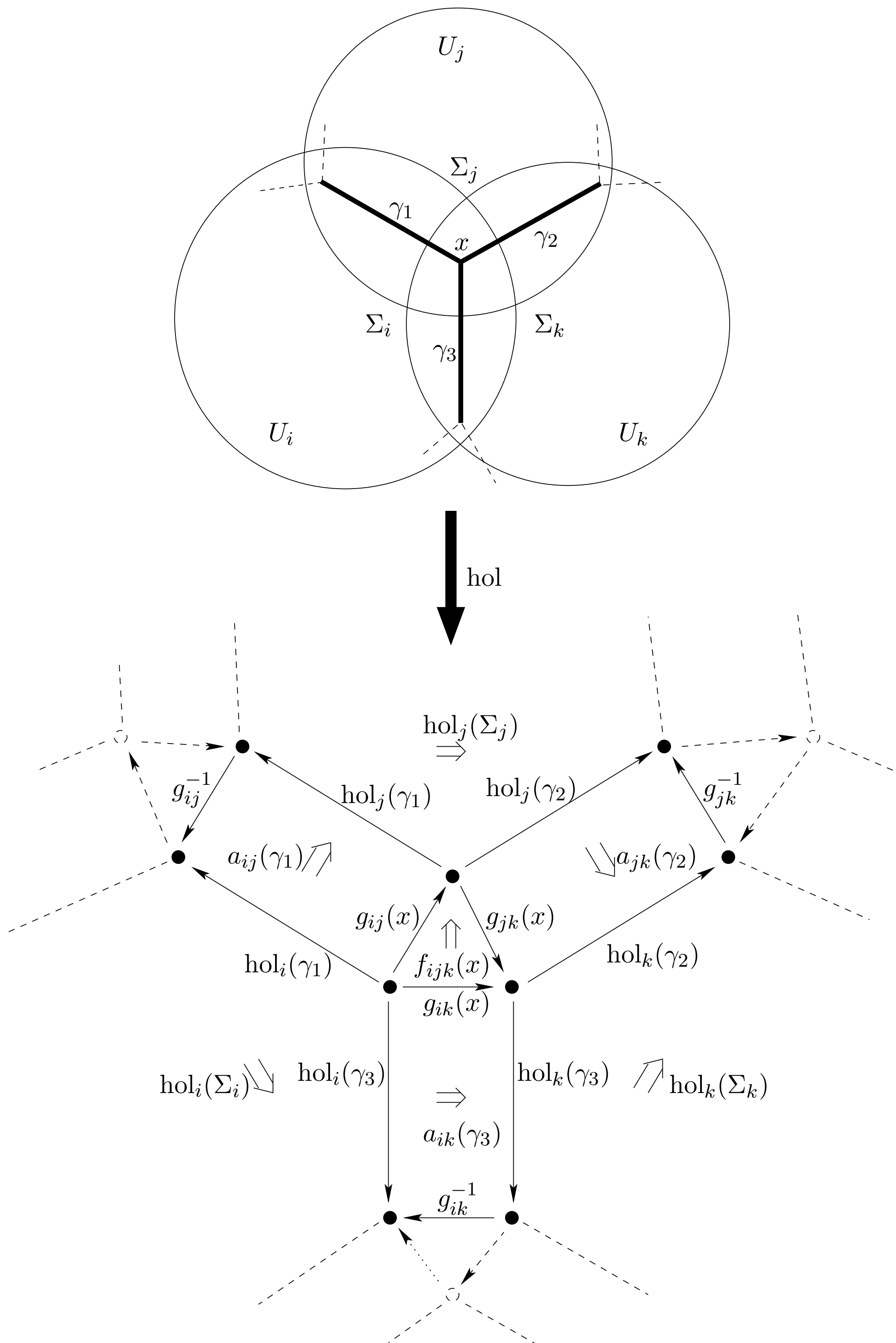
Applying this global 2-holonomy 2-functor hol

* to the 2-commuting diagram (12.4) in $\mathcal{P}_2^C(\mathcal{U})$ yields the transition law (11.13)
* to the 2-commuting diagram (12.3) yields the modification (11.14) further discussed in §11.4.3 (p.266)
* to the tetrahedron (12.2) in $C_2(\mathcal{U})$ yields the tetrahedron law (11.15).

Hence the global 2-holonomy 2-functor encodes precisely the information of a 2-bundle with 2-connection and 2-holonomy as defined in §11.3.2 (p.257).

In complete analogy to how we proceeded before for ordinary bundles, the 2-functor hol allows to compute surface holonomy by "lifting" a given surface in M to a 2-morphism in $\mathcal{P}_2(\mathcal{U})$ and applying hol on that.

This procedure is indicated by the following figure.



That the global holonomy constructed this way is indeed well defined (invariant under gauge transformations) follows again, as for the case of 1-bundles discussed before, from the fact that gauge transformations are nothing but (pseudo-)natural transformations of the global 2-holonomy 2-functor. This is discussed in the following §12.2.3. An analysis of the gauge invariance of the global 2-holony depicted in this figure from a slightly different point of view is given in §12.2.4 (p.299).

It is easy to see that in the abelian case the above figure encodes precisely the well-known concept of surface holononmy in abelian gerbes (and hence the proper action functional for strings in Kalb-Ramond backgrounds) as described in [42, 48, 45] and summarized for instance in [23].

For consider the case where the strict structure 2-group is given by the crossed module $\mathcal{G} = (G = 1, H = U(1), \alpha = \text{trivial}, t = \text{trivial})$. Then $\mathrm{hol}_i(\Sigma_i)$ is simply the exponetiated integral of B_i over Σ_i, $a_{ij}(\gamma)$ is (according to (11.27), p. 266) simply the line holonomy of a_{ij} along γ and the composition of all the 2-group elements in the above figure simply amounts to multiplying all these elements of $U(1)$. This is precisely the procedure discussed in the above mentioned references.

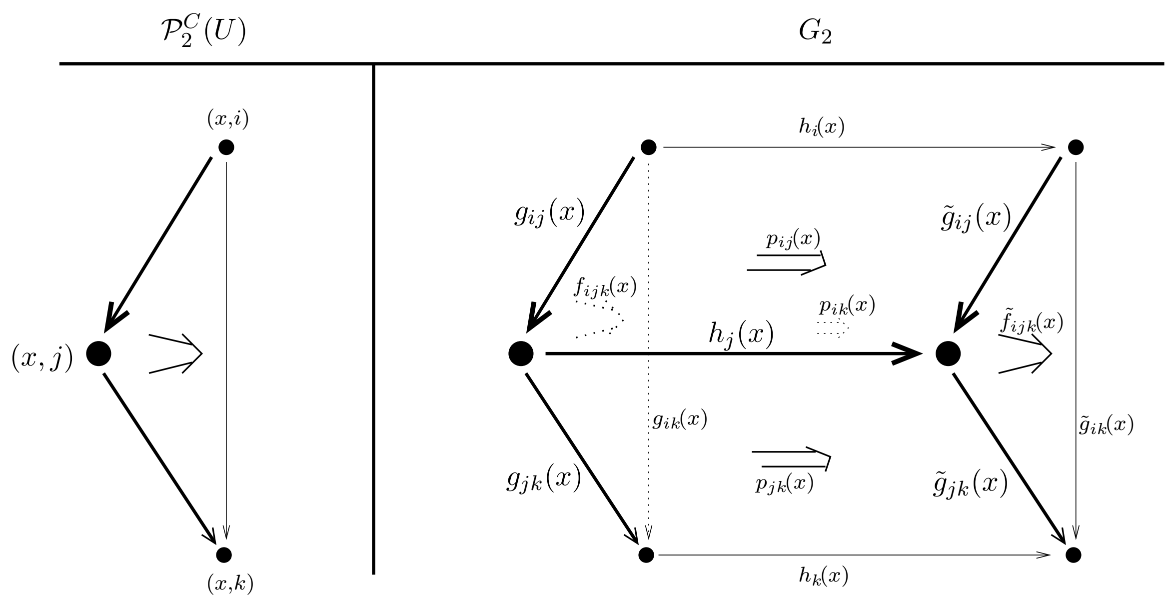
12.2.3 Gauge Transformations

What is very convenient about the abive formulation, where all the information about a 2-bundle with 2-connection is encoded in a single 2-holonomy 2-functor on a Čech -extended path 2-groupoid, is that, as was the case analogously for ordinary bundles before, the gauge transformations of the 2-bundle arise simply as natural pseudo-isomorphisms between two such 2-functors.

The effect of a gauge transformation

$$\mathrm{hol} \to \tilde{\mathrm{hol}}$$

on the transition functions g_{ij} and f_{ijk} is given by the following naturality diagram:



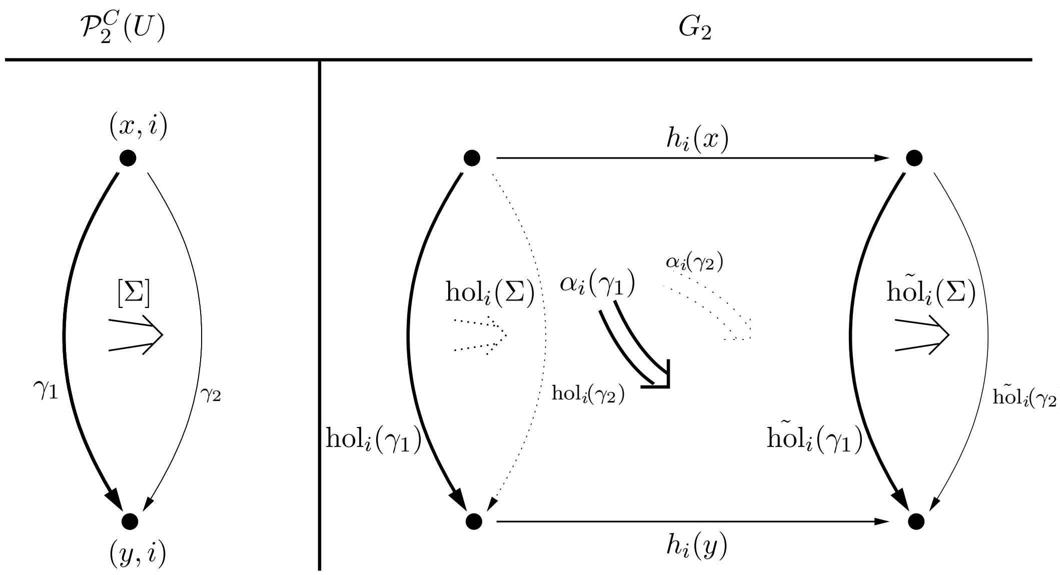
(12.5)

In terms of group elements this implies

$$t(p_{ij})\, g_{ij} = h_i \tilde{g}_{ij} h_j^{-1}$$

as well as a more unwieldy transformation equation for f_{ijk}.

The effect of the same gauge transformation on the 2-holonomy itself is given by another naturality diagram:



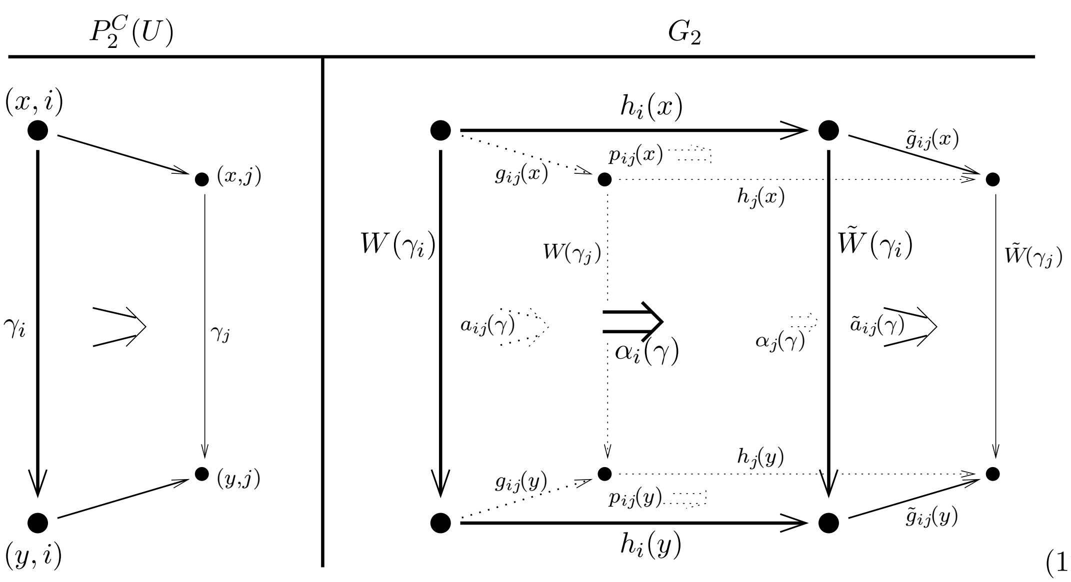
(12.6)

whose translation into formulas follows from proposition in §11.4.2 (p.264) and reads:

$$
\begin{aligned}
A_i &= h_i \tilde{A}_i h_i^{-1} + h_i \mathbf{d} h_i^{-1} - dt(\alpha_i) \\
B_i &= \alpha(h_i)\Big(\tilde{B}_i\Big) + \mathbf{d}_{A_i}\alpha_i + \alpha_i \wedge \alpha_i \,,
\end{aligned}
$$

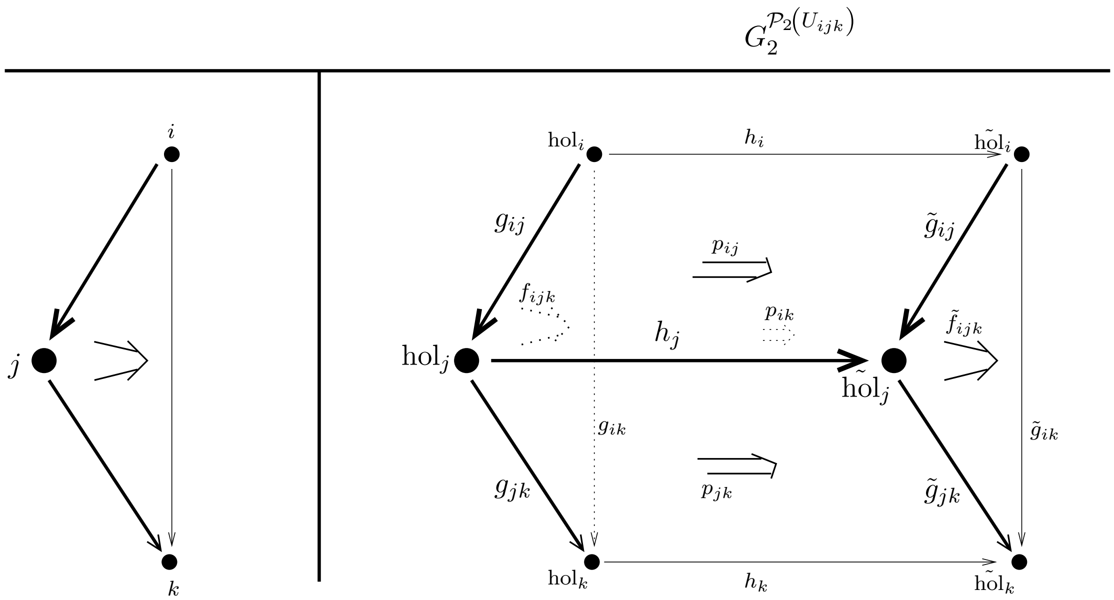
Here we have again used the same symbol to denote the 2-group morphism α_i and the 1-form it comes from.

Finally, the effect of the same gauge transformation on the transitions a_{ij} of the connection is given by this naturality diagram:



(12.7)

These are the (pseudo-)natural transformations of the global 2-holonomy 2-functor. Note that this can alternatively be understood as a (pseudo-)natural transformation of the simplicial map Ω defining a 2-bundle with 2-holonomy (as depicted in figure 7, p. 21 and detailed in §11.3.2 (p.257)), which assigns 2-holonomy 2-functors hol_i to patches U_i, assigns pseudo-natural transformations $\mathrm{hol}_i \xrightarrow{g_{ij}} \mathrm{hol}_j$ between these to double overlaps U_{ij} and assigns modifications $g_{ik} \xrightarrow{f_{ijk}} g_{ij} \circ g_{jk}$ to triple overlaps U_{ijk}. A transformation of such a map Ω looks like this:



Here $G_2^{\mathcal{P}_2(U_{ijk})}$ is the (2-)functor (2-)category (*cf.* §4.3.2.1 (p.81)) of 2-holonomy 2-functors form the 2-path 2-groupoid $\mathcal{P}_2(U_{ijk})$ of surfaces in the triple overlap U_{ijk} to the structure 2-group G_2.

The existence of

$$\mathrm{hol}_i \xrightarrow{h_i} \tilde{\mathrm{hol}}_i$$

is equivalent to diagram (12.6), the existence of

$$g_{ij} \xrightarrow{p_{ij}} h_i \circ \tilde{g}_{ij} \circ h_j^{-1}$$

is given by diagram (12.7) and the 2-commutativity is given by diagram (12.5).

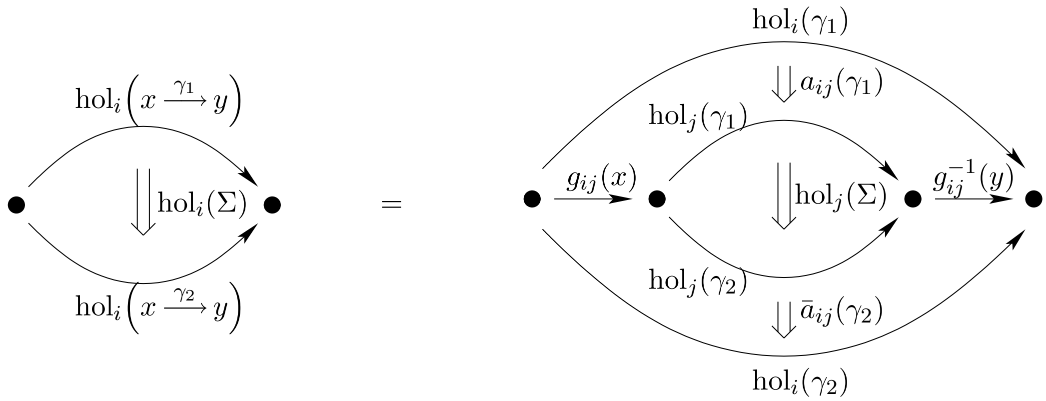
12.2.4 More Details

The reader might complain that the introduction of 2-connections with 2-holonomy in 2-bundles in §11.3.2 (p.257) does not obviously follow the categorification dictionary advertized in §1.1.4 (p.14). But in fact it does. Spelling this out in a little detail helps elucidate the nature of the more concise definitions in terms of pseudo-natural transformations of 2-functors that we have discussed above.

To begin with, consider the *equation* which describes the transition of an ordinary holonomy of a path $x \xrightarrow{\gamma} y$ in an ordinary (1-)bundle from patch U_i to patch U_j:

$$\mathrm{hol}_i(\gamma) = g_{ij}(x) \cdot \mathrm{hol}_j(\gamma) \cdot g_{ij}^{-1}(y) \; .$$

This is an equation between group element valued functions, where hol_i is regarded as a function on path space $P(U_i)$ and where $\cdot$ denotes the ordinary group product operation. Following the dictionary in §1.1.4 (p.14) we are to replace this by a natural isomorphism between 2-group-valued functors, where on the right the product is to become the product functor in the 2-group. Restricting to the special case that the categorification of $g_{ij}(x)$ are identity morphisms as in (12.9) (p. 304), this natural isomorphism is expressed as follows:



(12.8)

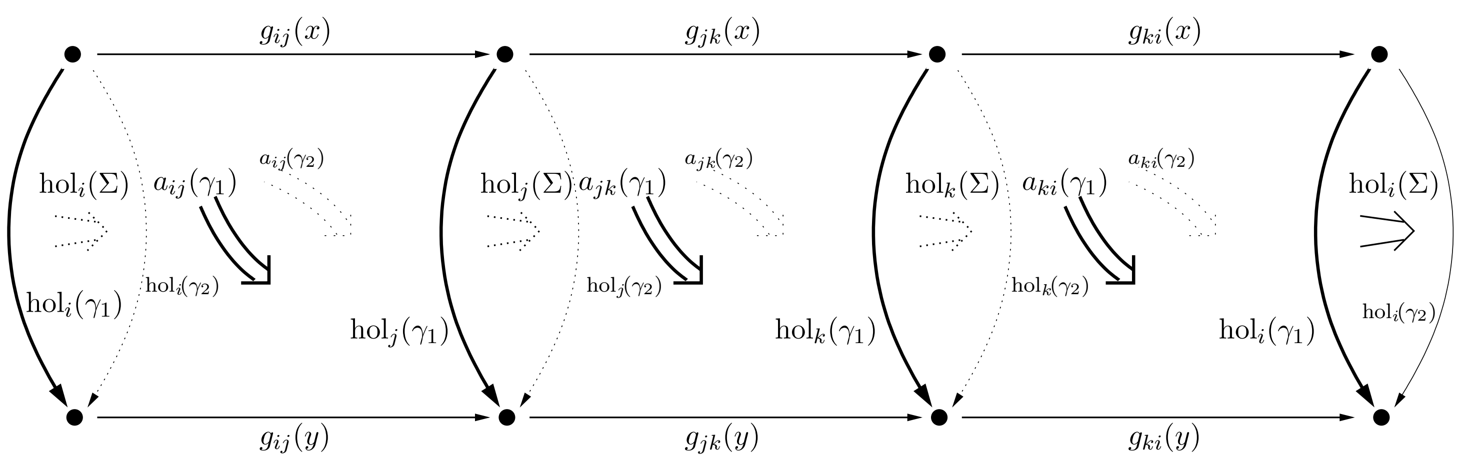
Here $\bar{a}_{ij}$ is the inverse of the morphism a_{ij}, which encodes the natural transformation. Note how the "horizontal conjugation" by g_{ij} is accompanied now by a "vertical conjugation" by a_{ij}.

But this is nothing but the 2-commutativity of the diagram (11.13) (p. 259), which expressed the existence of a pseuo-natural transformation between 2-holonomy 2-functors.

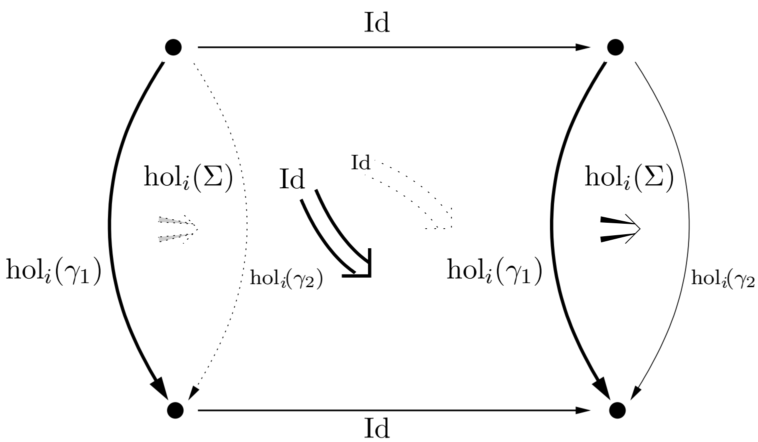
12.2.4.1 Transition on Triple Overlaps. From this perspective, the diagram (11.14) (p. 259) arises as a *coherence law* for the above natural isomorphism. Namely consider a transition

$$U_i \to U_j \to U_k \to U_i$$

from U_i to *itself* using a_{ij}, a_{jk} and a_{ki}, and demand that the result be the identity transformation. By the above, this amounts to demanding that this diagram

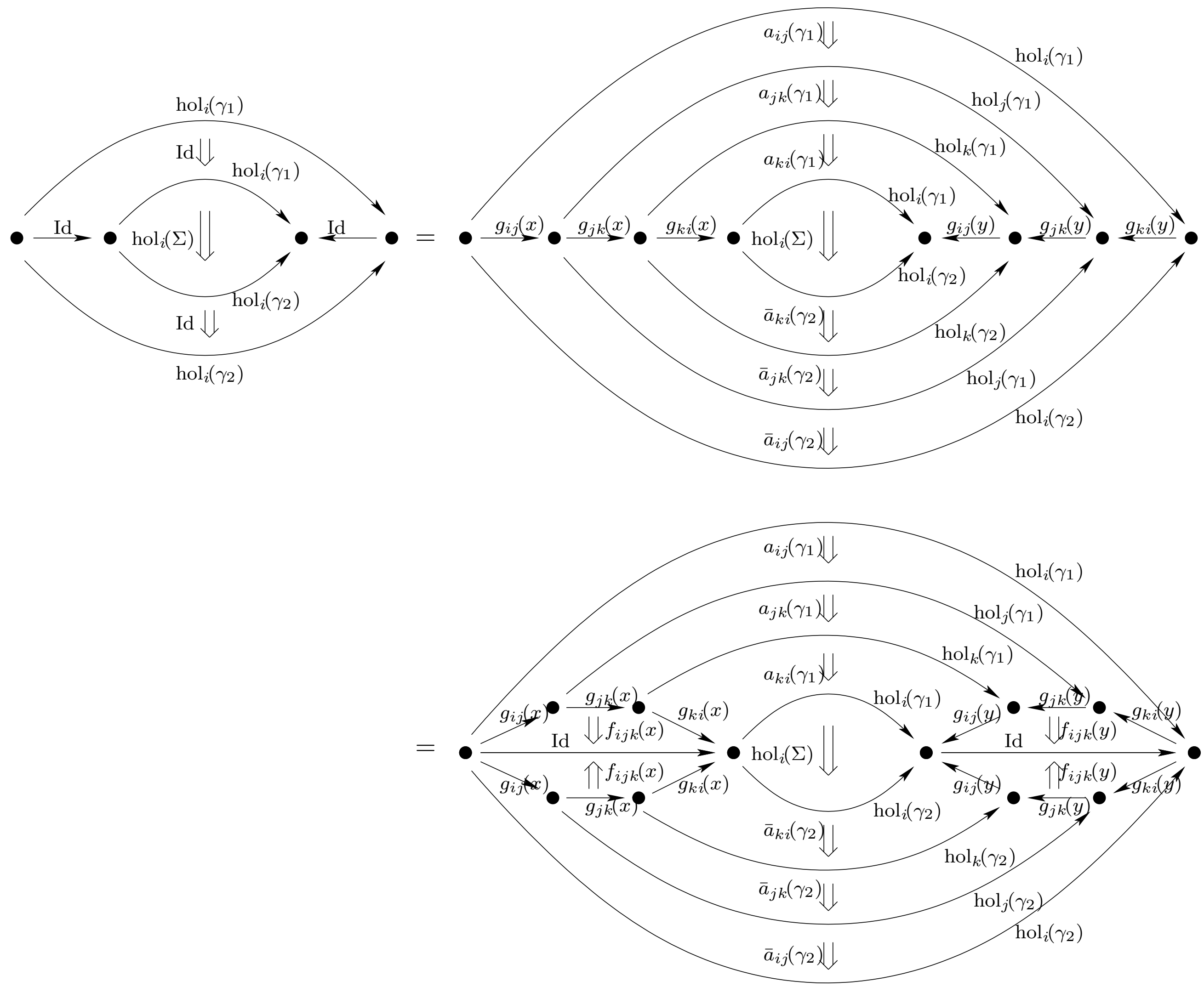


equals this diagram:



We want to show that this equality gives the 2-commutativity of the diagram (11.14) (p. 259).

For this purpose it is convenient to first redraw these cyclinder-like diagrams in planar form as follows:



The first equality sign here expresses the above cylindrical diagram. The point here is the step after the second equality sign, where two pairs of mutually cancelling f_{ijk} 2-morphisms have been inserted in order to obtain the two horizontal identity 1-morphisms. (Note that these are really defined only up to a 2-morphism $\mathrm{Id} \to \mathrm{Id}$, given by an element $h \in \ker t \subset H$. A nontrivial h here would give the transition law for a *3-bundle*.)

By comparison one sees that the upper half and the lower half of the diagram after the second equality sign are just a planar version of the diagram discussed in §11.4.3 (p.266). The equality of the above two cyclindrical diagrams is equivalent to the 2-commutativity of this diagram. Hence we can interpret the existence of the modification (11.14) in a 2-bundle with 2-holonomy as a coherence law on the transition on double overlaps.

As a byproduct, this tells us that equation (11.18) (p. 261) between the differential

forms encoding these transitions can be understood as equating the triple application of equations (11.17) with the identity transformation.

In order to see this, begin by writing down the transition law for the connection 1-form from U_i to U_j:

$$A_i = g_{ij} A_j (g_{ij})^{-1} + g_{ij}(\mathbf{d}(g_{ij})^{-1}) - dt(a_{ij})$$

This expresses A_i in terms of A_j. Now use the same transition law but from U_j to U_k in order to express the A_j in this formula in terms of A_k:

$$\cdots = g_{ij} \underbrace{\left(g_{jk} A_k g_{jk}^{-1} + g_{jk}(\mathbf{d} g_{jk}^{-1}) - a_{jk}\right)}_{=A_j} (g_{ij})^{-1} + g_{ij}(\mathbf{d}(g_{ij})^{-1}) - dt(a_{ij})$$

Finally, express A_k in terms of the original A_i:

$$\cdots = g_{ij} \underbrace{\left(g_{jk} \underbrace{\left(g_{ki} A_i g_{ki}^{-1} + g_{ki}(\mathbf{d} g_{ki}^{-1}) - dt(a_{ki})\right)}_{A_k} (g_{jk})^{-1} g_{jk}(\mathbf{d} g_{jk}^{-1}) - dt(a_{jk}) \right)}_{=A_j} (g_{ij})^{-1}$$

$$+ g_{ij}(\mathbf{d}(g_{ij})^{-1}) - dt(a_{ij})$$

After multiplying out the brackets this reads

$$\begin{aligned} A_i \;=\; & (g_{ij} g_{jk} g_{ki}) A_i (g_{ij} g_{jk} g_{ki})^{-1} + (g_{ij} g_{jk} g_{ki}) \mathbf{d} (g_{ij} g_{jk} g_{ki})^{-1} \\ & - dt(a_{ij}) - g_{ij} dt(a_{jk})\,(g_{ij})^{-1} - g_{ij} g_{jk} dt(a_{ki})\,(g_{ij} g_{jk})^{-1} \end{aligned}$$

Using the relation (11.4)

$$t(f_{ijk})\, g_{ik} = g_{ij} g_{jk}$$

this is simplified to

$$\begin{aligned} A_i \;=\; & A_i + t(h_{ijk}) \left[A_i, t(h_{ijk})^{-1}\right] + t(h_{ijk})\, \mathbf{d} \left(t(h_{ijk})^{-1}\right) \\ & - dt(a_{ij}) - g_{ij} dt(a_{jk})\,(g_{ij})^{-1} - t(h_{ijk})\, g_{ik}\, dt(a_{ki})\; g_{ik}(t(h_{ijk}))^{-1}\,. \end{aligned}$$

Finally we can factor out the action of dt:

$$A_i \;=\; A_i + dt\Big(h_{ijk} d\alpha(A_i)\Big(h_{ijk}^{-1}\Big) + h_{ijk} \mathbf{d} h_{ijk}^{-1} - a_{ij} - \alpha(g_{ij})(a_{jk}) - h_{ijk}\, \alpha\big(g_{ik}^{1}\big)(a_{ki})\; h_{ijk}^{-1}\Big)\,.$$

It follows that the term in brackets has to be in the kernel of dt, i.e.

$$h_{ijk} d\alpha(A_i)\Big(h_{ijk}^{-1}\Big) + h_{ijk} \mathbf{d} h_{ijk}^{-1} - a_{ij} - g_{ij}(a_{jk}) - h_{ijk}\, d\alpha\big(g_{ik}^{1}\big)(a_{ki})\; h_{ijk}^{-1} \;=\; -\alpha_{ijk}$$

with $\alpha_{ijk} \in \ker(dt)$. This can be simplified a little further: For $j = k$ this equation reduces to

$$a_{ik} + \alpha(g_{ik})(a_{ki}) \;=\; 0\,.$$

Reinserting this result yields

$$h_{ijk} d\alpha(A_i)\Big(h_{ijk}^{-1}\Big) + h_{ijk} \mathbf{d} h_{ijk}^{-1} - a_{ij} - g_{ij}^{1}(a_{jk}) + h_{ijk} a_{ik} h_{ijk}^{-1} \;=\; -\alpha_{ijk}\,.$$

Here α_{ijk} comes from the freedom to insert morphisms $\mathrm{Id} \to \mathrm{Id}$, mentioned above, which corresponds to freedom present in 3-bundles. Setting $\alpha_{ijk} = 0$ yields the promised equation (11.18).

12.2.4.2 Gauge Transformations In the same vein we can also understand the gauge transformations of 2-holonomy in 2-bundles that were discussed in §12.2.3 (p.297).

All we need is a categorification of the *equation*

$$\tilde{g}_{ij} = h_i \cdot g_{ij} \cdot h_j^{-1}\,,$$

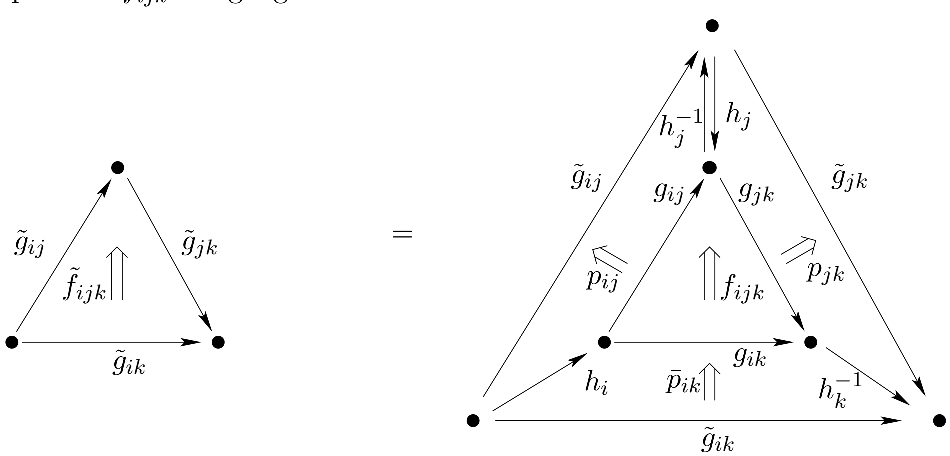
which expresses the gauge transformed transition function $\tilde{g}_{ij}$ in terms of the original one for ordinary bundles. As in (12.9) (p. 304) this gives a natural isomorphism

$$\begin{tikzcd} \bullet \arrow[r, bend left=40, "h_i g_{ij} h_j^{-1}"{name=U}] \arrow[r, bend right=40, "\tilde{g}_{ij}"'{name=D}] & \bullet \arrow[Rightarrow, from=U, to=D, "p_{ij}"] \end{tikzcd}$$

such that we have this naturality diagram:

$$\begin{array}{c|c} U_{ij} & \mathcal{G} \\ \hline \begin{tikzcd} x \arrow[d, "\mathrm{Id}"'] \\ x \end{tikzcd} & \begin{tikzcd} g_{ij}(x) \arrow[r, "p_{ij}(x)"] \arrow[d, "\mathrm{Id}"'] & h_i \cdot g_{ij} \cdot h_j^{-1}(x) \arrow[d, "\mathrm{Id}"] \\ g_{ij}(x) \arrow[r, "p_{ij}(x)"'] & h_i \cdot g_{ij} \cdot h_j^{-1}(x) \end{tikzcd} \end{array} \tag{12.9}$$

This implies for f_{ijk} the gauge transformation law



The equality here expresses precisely the 2-commutativity of the naturality diagram (12.5) (p. 297).

Furthermore, the gauge transformation equation for the holonomy

$$\tilde{\mathrm{hol}}_i(\gamma) = h_i(x) \cdot \mathrm{hol}_i(\gamma) \cdot h_i^{-1}(x)$$

is categorified precisely as in (12.8) (p. 300, recall the discussion there) and yields

$$
\begin{tikzcd}
\bullet \arrow[r, bend left=50, "\tilde{\mathrm{hol}}_i(\gamma_1)"] \arrow[r, bend right=50, "\tilde{\mathrm{hol}}_i(\gamma_2)"'] \arrow[r, phantom, "\Downarrow \tilde{\mathrm{hol}}_i(\Sigma)"] & \bullet
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\bullet \arrow[r, "h_i(x)"] \arrow[rrr, bend left=60, "\tilde{\mathrm{hol}}_i(\gamma_1)"] \arrow[rrr, bend right=60, "\tilde{\mathrm{hol}}_i(\gamma_2)"'] & \bullet \arrow[r, bend left=50, "\mathrm{hol}_i(\gamma_1)"] \arrow[r, bend right=50, "\mathrm{hol}_i(\gamma_2)"'] \arrow[r, phantom, "\Downarrow \mathrm{hol}_i(\Sigma)"] \arrow[r, phantom, yshift=12ex, "\Downarrow \alpha_i(\gamma_1)"] \arrow[r, phantom, yshift=-12ex, "\Downarrow \alpha_i(\gamma_2)"] & \bullet \arrow[r, "h_i^{-1}(y)"] & \bullet
\end{tikzcd}
$$

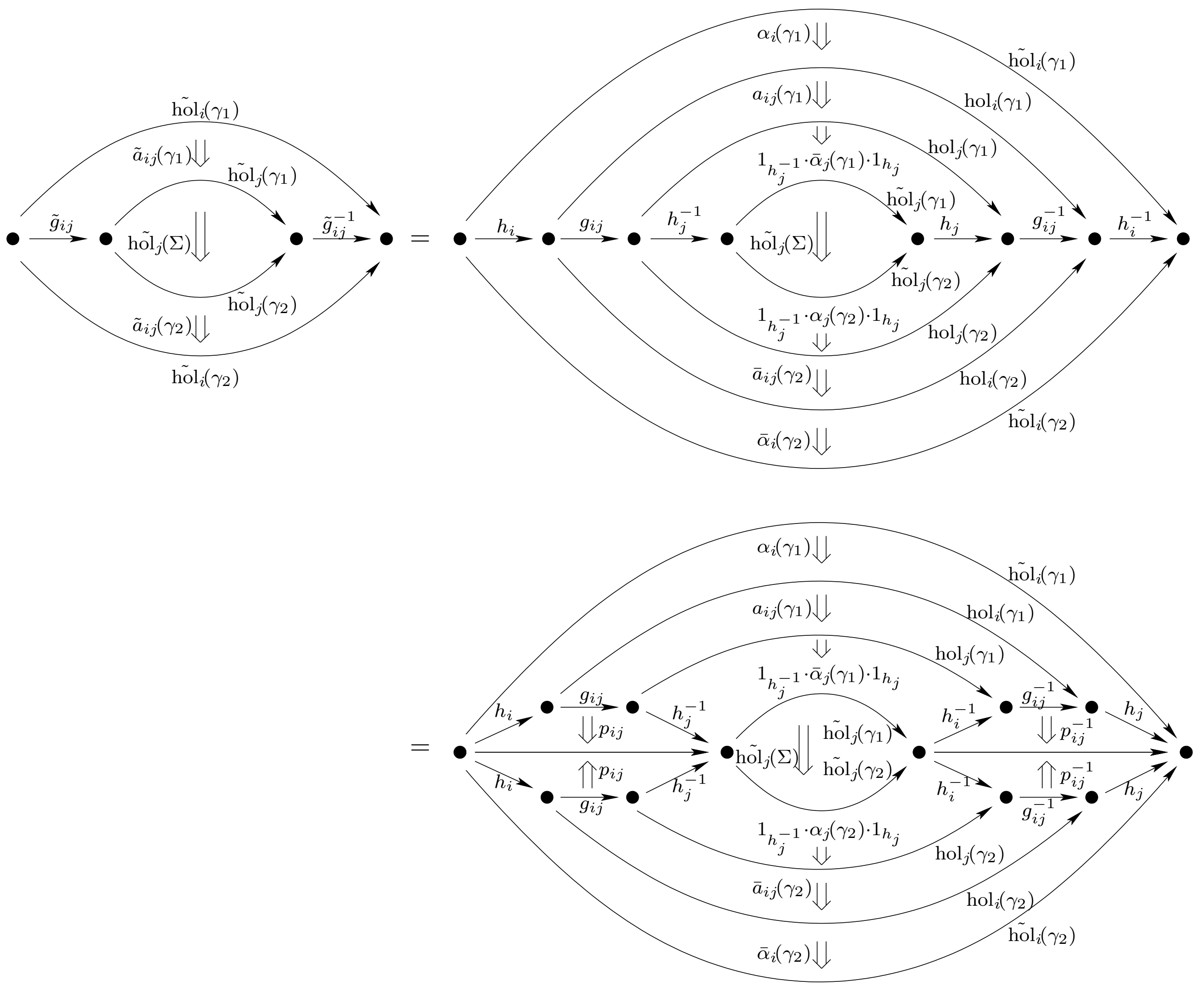
This yields the 2-commutativity of (12.6) (p. 298).

What requires, once again, a little more work is the analogous discussion for the a_{ij}

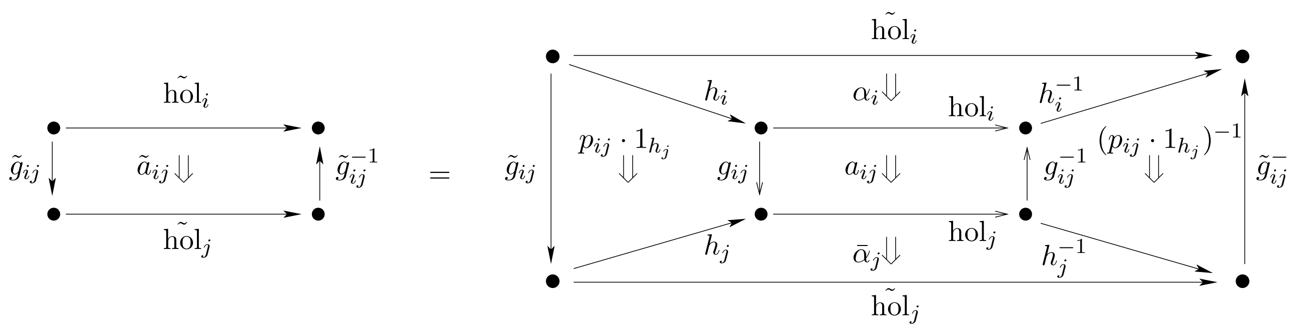
$$
\begin{tikzcd}
\bullet \arrow[r, "\mathrm{hol}_i\left(x \xrightarrow{\gamma} y\right)"] \arrow[d, "g_{ij}(x)"'] \arrow[dr, phantom, "\Downarrow a_{ij}\left(x \xrightarrow{\gamma} y\right)"] & \bullet \\
\bullet \arrow[r, "\mathrm{hol}_j\left(x \xrightarrow{\gamma} y\right)"'] & \bullet \arrow[u, "g_{ij}^{-1}(y)"']
\end{tikzcd}
\tag{12.10}
$$

giving the derivation of (12.7) (p. 298).

This is obtained by considering a transition first in one gauge and then in the other:



From this one reads off the gauge transformation law of the a_{ij} as



And indeed, this equation expresses the 2-commutativity of (12.7).

This way, all the gauge transformation laws discussed in §12.2.3 (p.297) are reobtained.

12.2.4.3 Gauge Invariance of Global 2-Holonomy. In the same pedestrian way we can now analyze the gauge invariance of global 2-holonomy, which in the formulation of §12.2 (p.291) is nothing but 2-functoriality of the global 2-holonomy 2-functor hol.

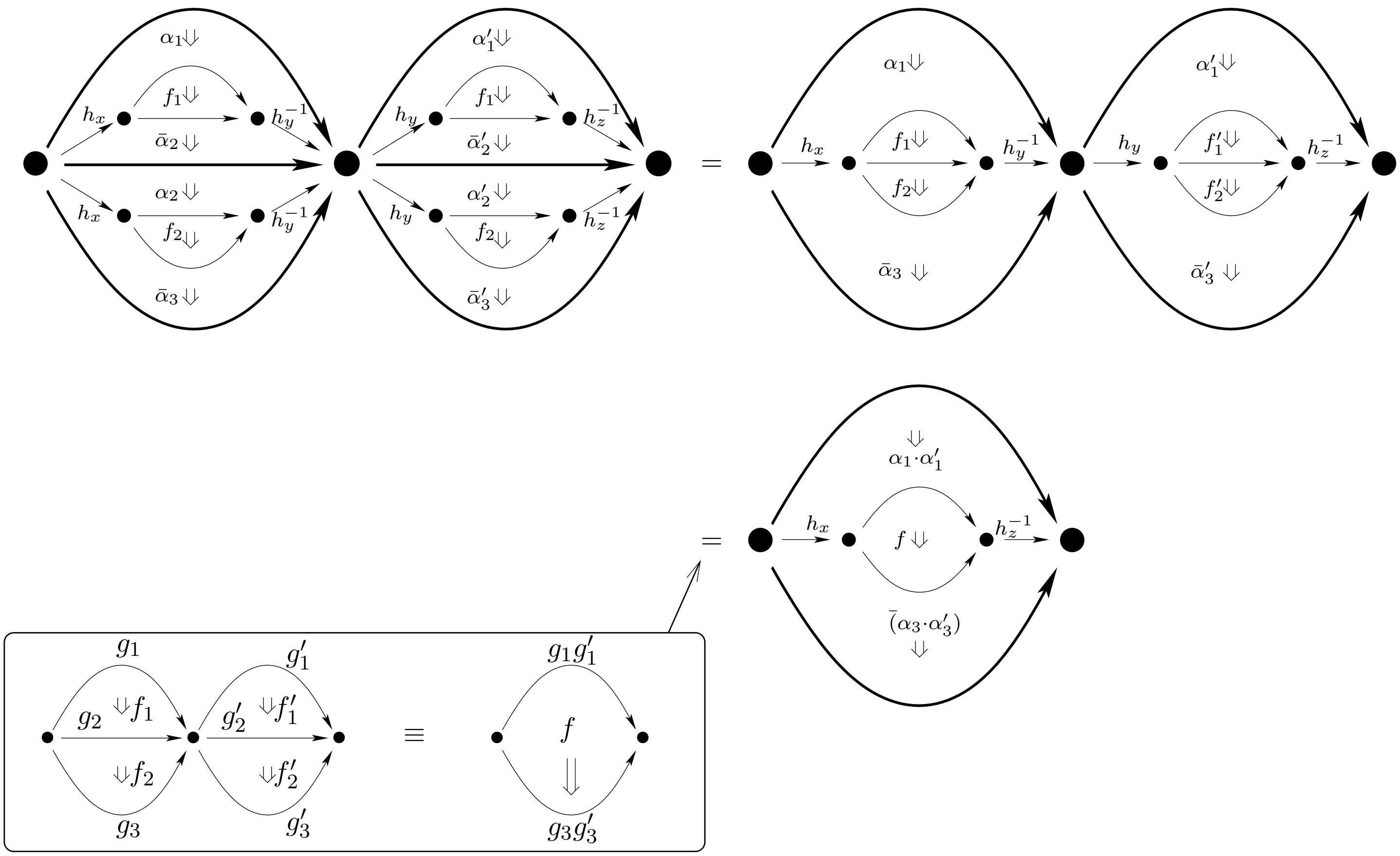
Given any closed surface in base space whose 2-holonomy is to be computed, we can triangulate it such a way that each face comes to lie in an element of the cover, each edge in a double overlap and each vertex in a triple overlap. We can always assume the graph of the triangulation to be trivalent. (If it is not we replace the problematic vertices by small circles of edges.)

The task is to assign 2-group elements to faces, edges and vertices of the triangularization such that the result of gluing them all together is independent of the choice of gauge (trivialization) as well as of the choice of cover and the choice of triangularization. For now we restrict attention on independence of the gauge choice.

It is clear that local 2-holonomies $\mathrm{hol}_i(\Sigma)$ must be assigned to faces Σ. The only candidate 2-group elements to be assigned to edges γ are $a_{ij}(\gamma)$ and the only candidate 2-group elements to be assigned to vertices x are $f_{ijk}(x)$.

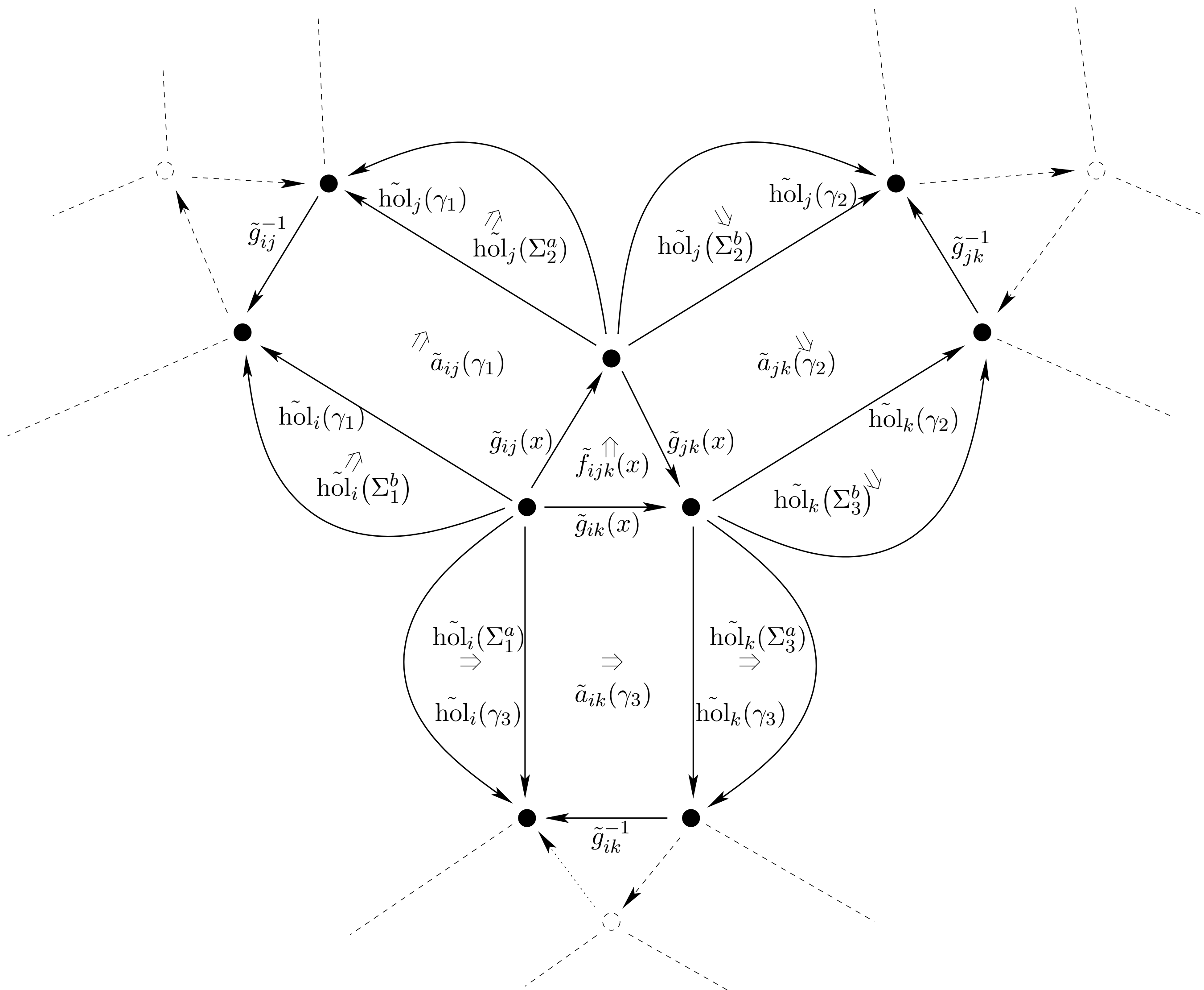
There is only one way to glue all these pieces consistently, and this is the way depicted in figure 8 (p. 24).

Before looking at the gauge invariance of this definition notice how "2-conjugations" (horizontal and vertical conjugation) respects the composition in the 2-group in the following sense:



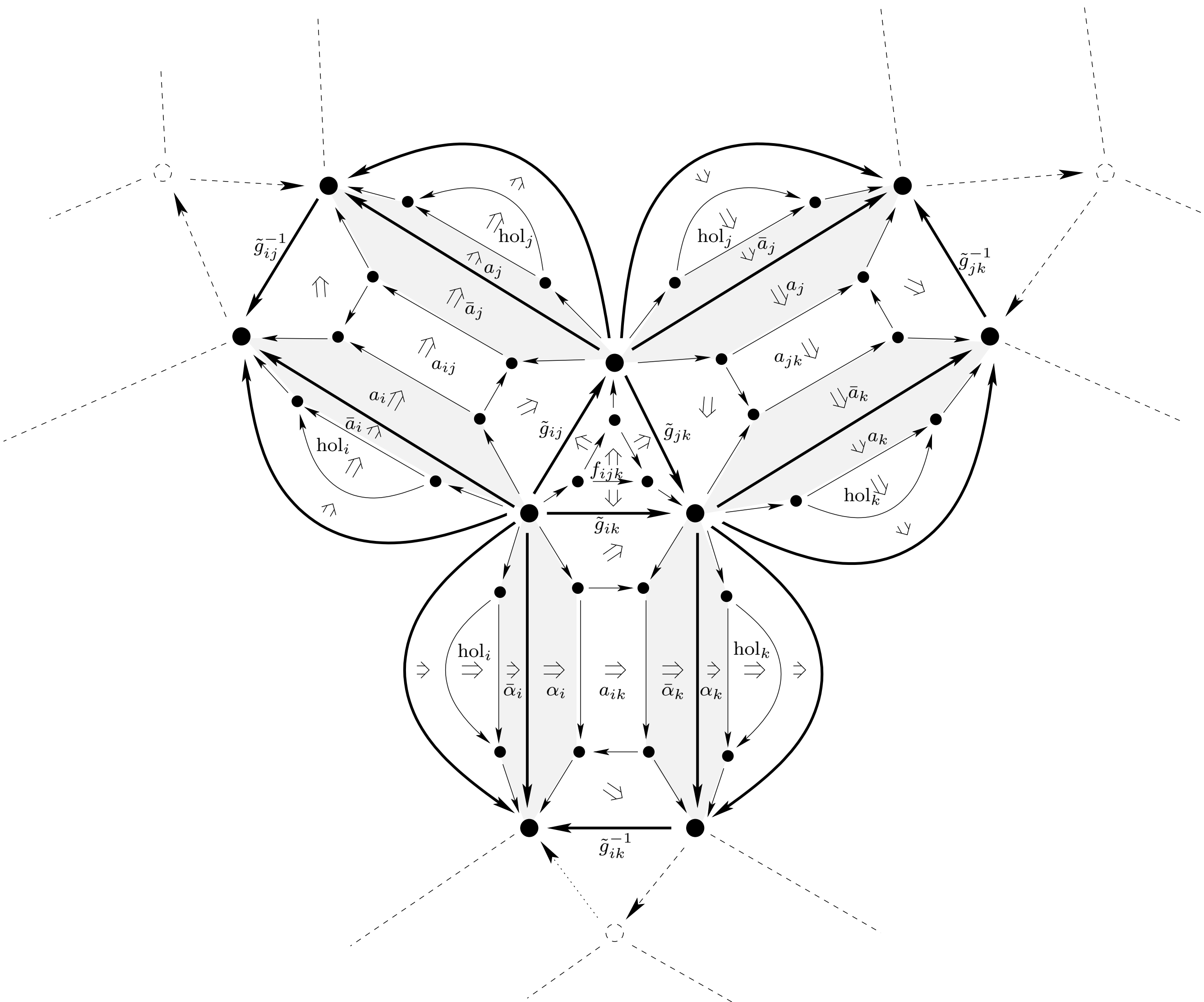
We display this rather simple fact in such a detail because it gives a good illustration of the way how in a composite diagram of gauge transformed quantities 2-conjugation operations mutually cancel and leave the diagrams in the original gauge behind. The demonstration of the gauge invariance of global 2-holonomy further below proceeds in this fashion.

Fix the gauge $\tilde{G}$ and let the surface holonomy in the vicinity of some vertex x be given by



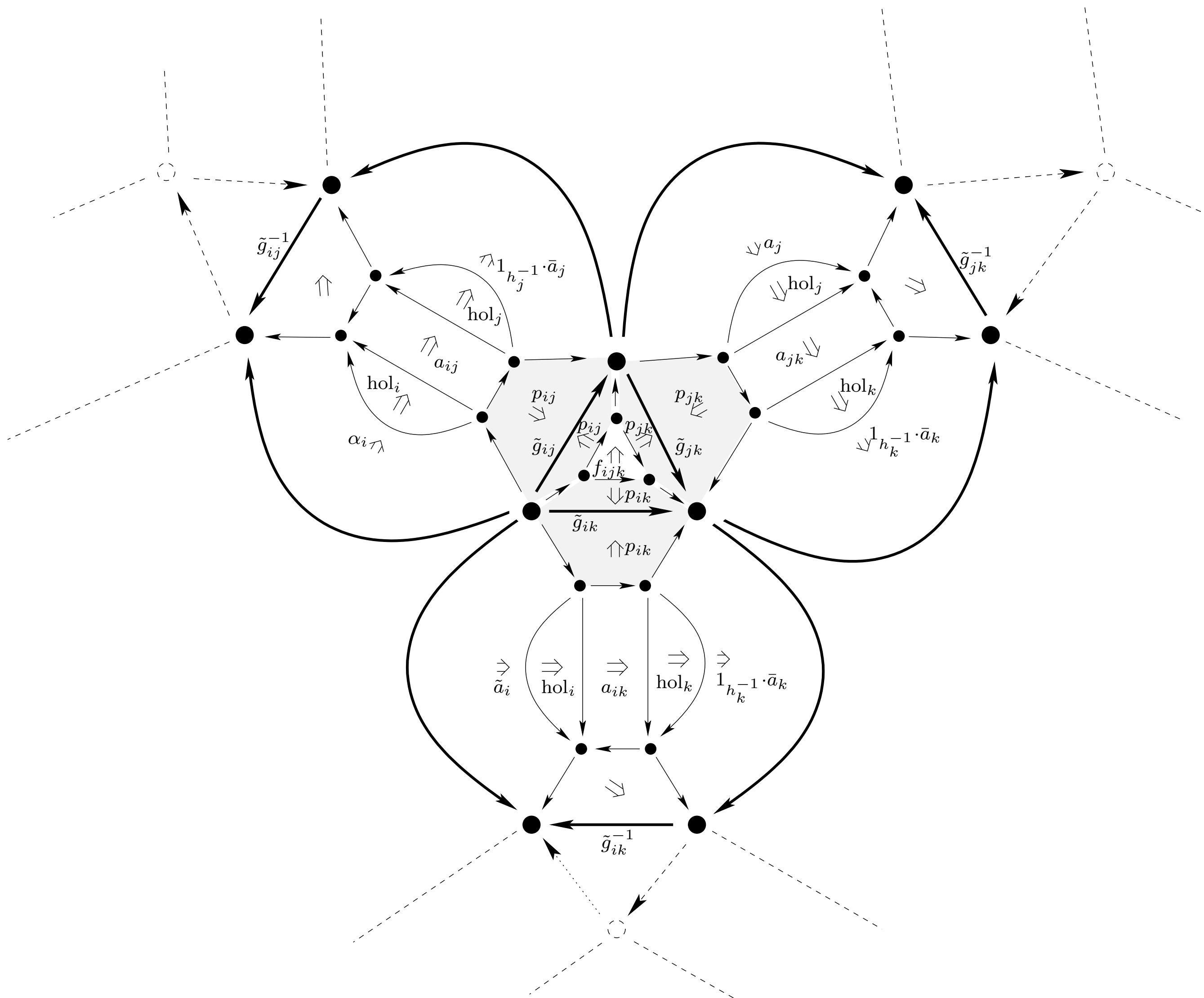
In this diagram the full local surface holonomies $\tilde{\mathrm{hol}}_i(\Sigma_1)$, $\tilde{\mathrm{hol}}_k(\Sigma_2)$, $\tilde{\mathrm{hol}}_j(\Sigma_3)$ are depicted only in terms of two surface sub-elements $\Sigma_i^a, \Sigma_i^b \subset \Sigma_i$ etc., respectively, adjacent to the edges meeting at the given vertex.

Now we insert into this diagram the equalities discussed in §12.2.4.2 (p.304), which re-express the diagrams in the gauge $\tilde{G}$ in terms of those in the gauge G:

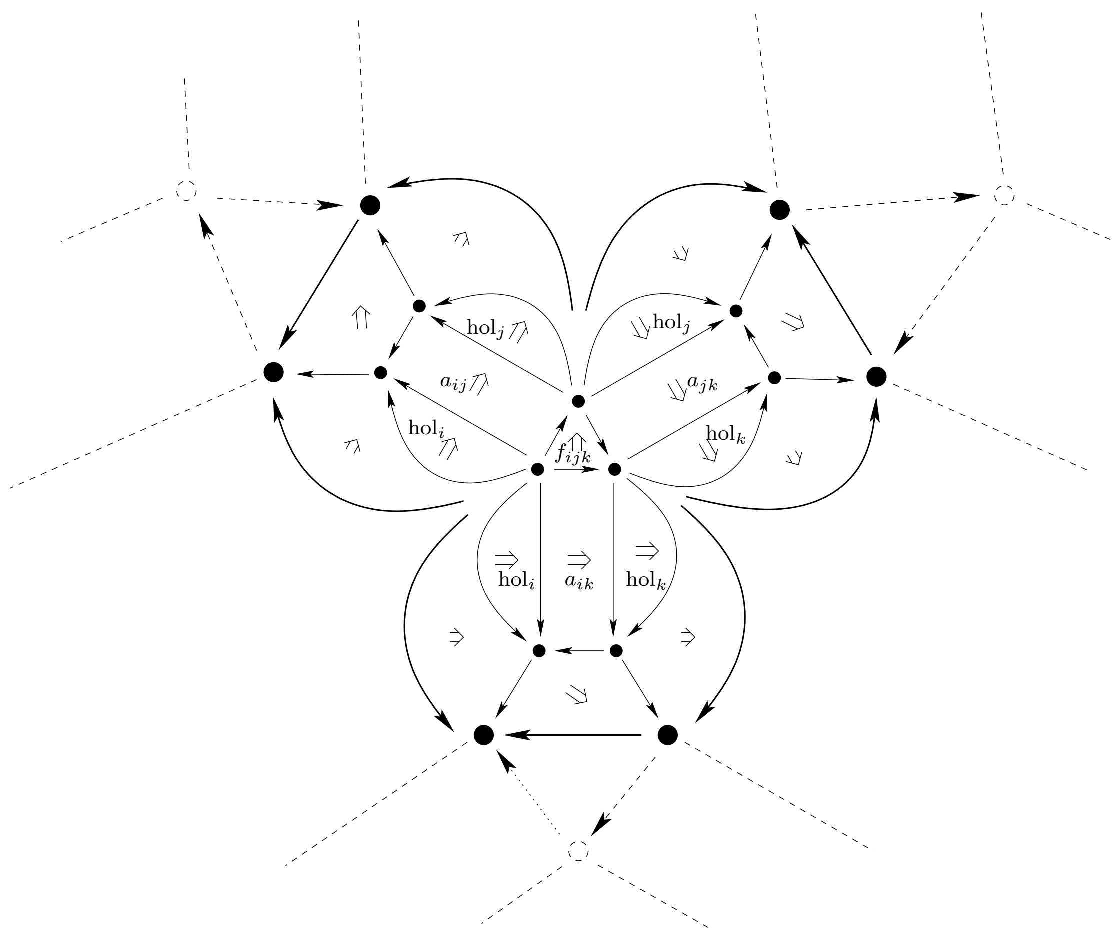


The result is that several 2-group elements are now adjacent which mutually cancel to unity. First of all we can cancel a_{ij} against $\bar{a}_{ij}$ and analogously for jk and ij. The respective identity 2-morphisms have been shaded in the diagram.

After removing them we are left with the following diagram:



We have also reversed the direction of some edges by whiskering, so that we can now cancel p_{ij} against $\bar{p}_{ij}$. When the shaded identity 2-morphisms are removed one obtains the following diagram



In the center of this diagram the surface holonomy in the gauge G has appeared. It is surrounded by 2-conjugations which cancel against the contributions from the other vertices.

This demonstrates the gauge invariance of our global 2-holonomy. In order to demonstrate the invariance under different choices of triangulations and of good covering, it is advisable to adopt a more sophisticated approach towards 2-holonomy, namely a more intrinsic one that makes use of *torsors* and does not require local trivializations. This is the content of §12.4 (p.314).

12.3 3-Bundles with 3-Connection

It is relatively straightforward to repeat the step from 1-bundles with 1-connection to 2-bundles with 2-connection again and again. The details will get more and more involved, but the general principle remains the same.

We will not enter a full discussion of 3-bundles here, but will want to emphasize the following easily accessible fact about 3-bundles which will have relevance for a good understanding of 2-bundles:

12.3.1 The Čech -extended 3-Path 3-Groupoid

From the above discussion it is clear that the Čech 3-Groupoid $C_3(\mathcal{U})$ and hence the Čech -extended path 3-groupoid $\mathcal{P}_3^C(U)$ of a locally trivialized 3-bundle contains 3-morphisms of the following kind:

$$
\begin{tikzcd}
(x,j) \arrow[r] \arrow[dr, phantom, "\Searrow", near start] & (x,k) \arrow[d] \arrow[dl, phantom, "\Downarrow", near start] \arrow[d, phantom, ""] & & (x,j) \arrow[r] \arrow[dr] \arrow[d, phantom, ""] & (x,k) \arrow[d] \arrow[dl, phantom, "\Swarrow", near start] \\
(x,i) \arrow[u] \arrow[ur] \arrow[r] & (x,l) \arrow[r, phantom, "\Rrightarrow"] & & (x,i) \arrow[u] \arrow[r] \arrow[ur, phantom, "\Downarrow", near start] & (x,l)
\end{tikzcd}
$$

These are 3-morphisms inside of each Čech -tetrahedron.

12.3.2 The global 3-Holonomy 3-Functor

Applying a holonomy 3-functor to these produces a 3-morphism λ_{ijkl} in a 3-group (*cf.* §10.7 (p.237)):

$$
\begin{tikzcd}
\bullet \arrow[r, "g_{jk}"] \arrow[dr, phantom, "\Searrow f_{ijk}", near start] & \bullet \arrow[d, "g_{kl}"] \arrow[dl, phantom, "f_{ikl}\Downarrow", near start] & & \bullet \arrow[r, "g_{jk}"] \arrow[dr, "g_{jl}"] & \bullet \arrow[d, "g_{kl}"] \arrow[dl, phantom, "f_{jkl}\Swarrow", near start] \\
\bullet \arrow[u, "g_{ij}"] \arrow[ur, "g_{ik}"] \arrow[r, "g_{il}"'] & \bullet \arrow[r, phantom, "\overset{\lambda_{ijkl}}{\Rrightarrow}"] & & \bullet \arrow[u, "g_{ij}"] \arrow[r, "g_{il}"'] \arrow[ur, phantom, "\Downarrow f_{ijl}", near start] & \bullet
\end{tikzcd}
$$

In terms of the 2-crossed module $(G, H, J, \alpha_1, \alpha_2, t_1, t_2) \simeq G_3$ of the structure 3-group (see §10.7.1 (p.237)) this says that

$$t_2(\lambda_{ijkl})\, f_{ijk} f_{ikl} = \alpha_1(g_{ij})(f_{jkl})\, f_{ijl}\,.$$

It is well known that $t_2(\lambda_{ijkl})$ satisfying this equation defines a class in Čech cohomology

$$[t_2(\lambda_{ijkl})] \in \check{H}^3(M) = H^4(M;\mathbb{Z})\ .$$

For the special case that the 3-group G_3 is an extension of the 2-group $\mathcal{P}_k G$ as described in §10.7.2 (p.240) it is a theorem in [179] that this class is the characteristic class $p_1/2$ of a G-bundle $E \to M$.[13]

This should mean that the obstruction to having a $\mathcal{P}_k G$-2- bundle on M lifting a principal G bundle on M is precisely this class. According to the discussion in §4.2 (p.68) this would mean that $\mathcal{P}_1\mathrm{Spin}(n)$-2-bundles have the right properties to describe the parallel transport of spinning strings.

Finally we note one fact about 3-connections with 3-holonomy in 3-bundles that generalizes the condition of vanishing fake curvature for 2-holonomy and which will be rederived in linearized form using nonabelian Deligne hypercohomology in §13.8 (p.369):

Proposition 12.1 *For G_3 a strict 3-group (cf. 10.7 (p.237)), $G_3 = (G, H, J, \alpha_{1,2}, t_{1,2})$, the 3-Holonomy functor over a patch U_i is specified by a set $\{A_i, B_i, C_i\}$ with*

$$\begin{aligned} A_i &\in \Omega^1(U_i, \mathfrak{g}) \\ B_i &\in \Omega^2(U_i, \mathfrak{h}) \\ C_i &\in \Omega^3(U_i, \mathfrak{j}) \end{aligned}$$

which has to satisfy the two consistency conditions

$$\begin{aligned} dt_1(B_i) + F_{A_i} &= 0 \\ dt_2(C_i) + \mathbf{d}_{A_i} B_i &= 0\,. \end{aligned}$$

This follows from similar consideration as in one dimension lower, using the fact [31] that for vanishing fake curvature the curvature on path space given by

$$\mathcal{F}_i = \oint_A (\mathbf{d}_{A_i} B_i)\ .$$

12.4 p-Functors from p-Paths to p-Torsors

So far we have constructed p-holonomy making use of good coverings of the base manifold. This has the advantage that this way p-holonomy can locally be defined as nothing but a p-functor to the structure p-group. But the resulting global p-holonomy p-functor defined

[13] I am grateful to Branislav Jurčo and Danny Stevenson for making me aware of this theorem.

by gluing these local p-functors by p-functor n-morphisms on $(n+1)$-fold overlaps is not *manifestly* insensitive to the choice of good covering.

The reason for this is that a given fiber of a principal G_p-p-bundle is, while isomorphic to G_p, not *canonically* isomorphic to G_p. It is a G_p-*p-torsor* rather than G_p itself.

A nice pedagogical introduction to torsors can be found in [193]. The precise definition of (1-)torsors and 2-torsors is stated for instance in Toby Bartels' paper on 2-bundles [36].

Hence a holonomy functor should really associate to any path in the base manifold a *torsor morphism* between the fibers above the endpoints of the path. More precisely, a p-holonomy p-functor should be a p-functor from p-paths in the base manifold to p-morphisms of p-torsors.

$$\mathrm{hol}\colon \mathcal{P}_p(M) \to G_p{-}p\mathrm{Tor}\,.$$

Even more precisely, since we would want hol to be a smooth p-functor between smooth p-categories, while $G_p - p\mathrm{Tor}$ does not have a smooth structure, we should have a p-functor

$$\mathrm{hol}\colon \mathcal{P}_p(M) \to \mathrm{Trans}_p(E)\ .$$

Here $\mathrm{Trans}_p(E)$ is the p-category of **p-transporters** in a principal G_p-p-bundle $E \to M$ over a categorically trivial base manifold M. Objects of $\mathrm{Trans}_p(E)$ are the fibers E_x of E for all $x \in M$, regarded as G_p-p-torsors, and n-morphisms in $\mathrm{Trans}_p(E)$ are their p-torsor n-morphisms.

Given any such p-functor to $\mathrm{Trans}_p(M)$ we can always forget about the smooth structure and regard it as a p-functor to $G_p{-}p\mathrm{Tor}$. For notational convenience, this is what we shall do in the following.

Indeed, this more intrinsic description of global p-holonomy is *equivalent* to the one we have been concentrating on up to this point. In §12.4.1 (p.315) we recall the well-known way how this equivalence works for 1-bundles. Then in §12.4.2 (p.321) the generalization to 2-bundles is spelled out.

12.4.1 1-Torsors and 1-Bundles with Connection and Holonomy

A principal G-bundle with connection over a base manifold M is specified by a functor

$$\mathrm{hol}\colon \mathcal{P}_1(M) \to G\text{–}\mathrm{Tor}\,.$$

We shall now choose any good covering $\mathcal{U} \to M$ of M, $\mathcal{U} = \bigsqcup_{i\in I} U_i$, and demonstrate how this single functor encodes local holonomy functors

$$\mathrm{hol}_i\colon \mathcal{P}_1(U_i) \to G$$

on each U_i, which are related on double overlaps by natural transformations $\mathrm{hol}_i \xrightarrow{g_{ij}} \mathrm{hol}_j$, as described in §3.3.1 (p.55). In the process of doing so, the procedure for computing global 1-holonomy in terms of the hol_i, as described at the end of §3.3.1 (p.55), drops out automatically.

For 1-bundles this is all rather simple and very well known. We find it worthwhile to restate these facts in order to make the generalization to 2-bundles in §12.4.2 (p.321) more accessible.

So let $\mathcal{U} \to M$ be a good covering of M, $\mathcal{U} = \bigsqcup_{i \in I} U_i$. When restricted to any of the U_i, the functor hol is naturally isomorphic to a local holonomy functor

$$\mathrm{hol}_i \colon \mathcal{P}_2(U_i) \to G\,.$$

Fix any such natural isomorphism

$$\mathrm{hol}|_{U_i} \xrightarrow{t_i} \mathrm{hol}_i\,.$$

It is specified by a naturality square

$$\begin{array}{c|c} \mathcal{P}_1(U_i) & G\text{–Tor} \\ \hline \begin{tikzcd} x \arrow[d, "{[\gamma]}"'] \\ y \end{tikzcd} & \begin{tikzcd} E_x \arrow[r, "t_i(x)"] \arrow[d, "\mathrm{hol}|_{U_i}(\gamma)"'] & G \arrow[d, "\mathrm{hol}_i(\gamma)"] \\ E_y \arrow[r, "t_i(y)"'] & G \end{tikzcd} \end{array}$$

Here we use the fact that G itself is a G-torsor and that its automorphisms in G–Tor correspond to (right)-multiplication with elements in G.

It follows that on double overlaps we have natural isomorphisms

$$\mathrm{hol}_i \xrightarrow{g_{ij}} \mathrm{hol}_j$$

given by this commuting diagram

$$\begin{tikzcd} & \mathrm{hol}|_{U_{ij}} \arrow[dr, "t_j(x)"] & \\ \mathrm{hol}_i \arrow[ur, "\bar{t}_i(x)"] \arrow[rr, "g_{ij}(x)"'] & & \mathrm{hol}_j \end{tikzcd}$$

.

Here $\bar{t}_i$ denotes the inverse of t_i. We write $\bar{t}_i$ because later on, when we categorify, t_i will be only weakly invertible (up to equivalence) and $\bar{t}_i$ will denote any of its weak inverses.

12.4.1.1 Line Holonomy in Terms of local Trivializations. Now consider the application of hol to any morphism $[\gamma] \in \mathcal{P}_1(M)$ that does not necessarily sit in a single patch U_i

$$\mathrm{hol}\left(\begin{tikzcd} x \arrow[rr, bend left, "{[\gamma]}"] & & y \end{tikzcd} \right) = \begin{tikzcd} E_x \arrow[rr, bend left, "\mathrm{hol}(\gamma)"] & & E_y \end{tikzcd} .$$

We can always split up $[\gamma]$ into pieces

$$\begin{tikzcd} x \arrow[rr, bend left, "{[\gamma]}"] & & y \end{tikzcd} = \begin{tikzcd} x \arrow[rr, bend left, "{[\gamma_{i_1}]}"] & & y_{i_1} \arrow[rr, bend left, "{[\gamma_{i_2}]}"] & & y_{i_2} \cdots \arrow[rr, bend left, "{[\gamma_{i_n}]}"] & & y \end{tikzcd}$$

such that $\gamma_{i_m} \in \mathcal{P}_2(U_{i_m})$, $\forall\, m$. Applying hol to that similarly yields

$$\mathrm{hol}\left(\begin{tikzcd} x \arrow[rr, bend left, "{[\gamma]}"] & & y \end{tikzcd} \right) = \begin{tikzcd} E_x \arrow[rr, bend left, "\mathrm{hol}(\gamma_{i_1})"] & & P_{y_{i_1}} \arrow[rr, bend left, "\mathrm{hol}(\gamma_{i_2})"] & & P_{y_{i_2}} \cdots \arrow[rr, bend left, "\mathrm{hol}(\gamma_{i_n})"] & & E_y \end{tikzcd} .$$

But at this point we can apply the commutativity of the above naturality square and set

$$\mathrm{hol}(\gamma_i) = t_i(x) \circ \mathrm{hol}_i(\gamma_i) \circ t_i^{-1}(y) \ .$$

This procedure is indicated in figure 11.

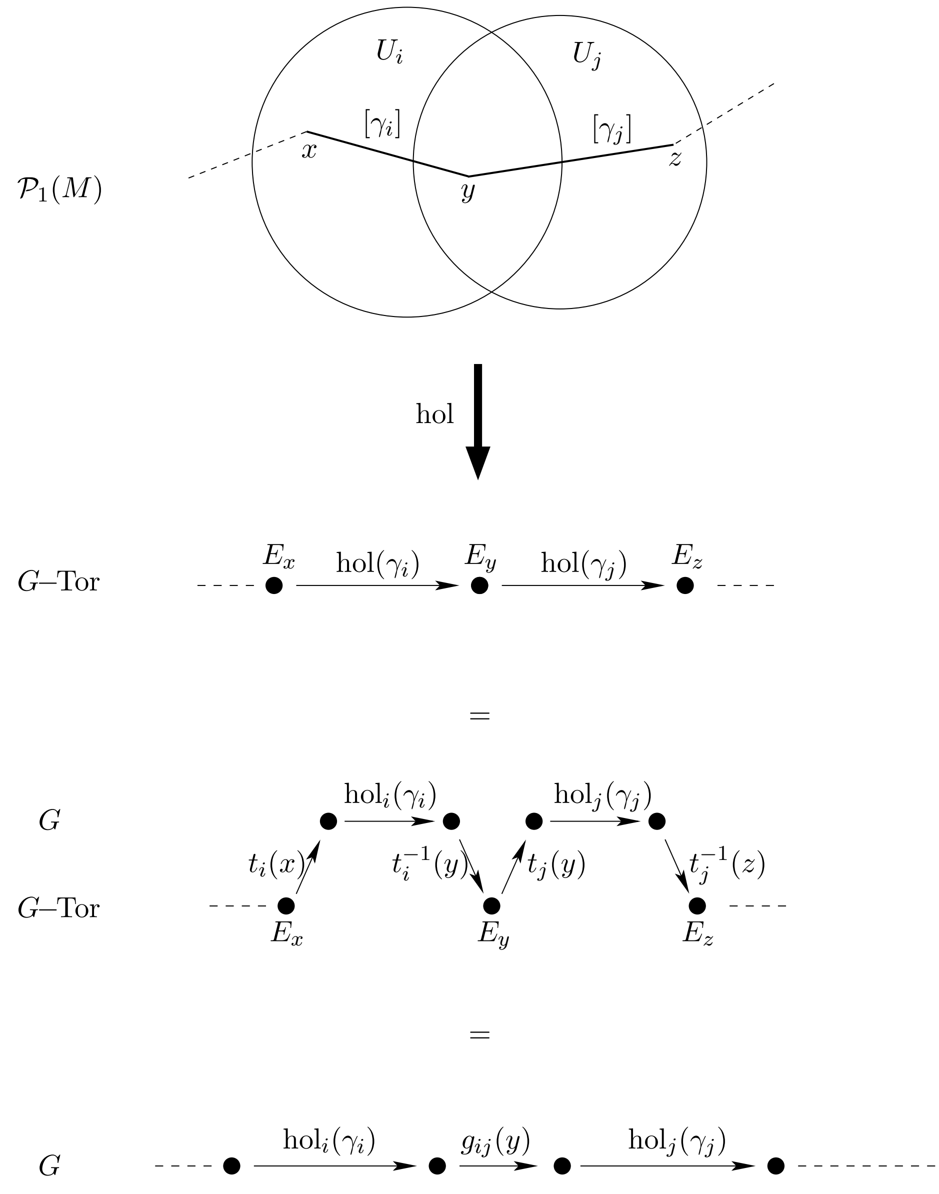


Figure 11: Global (1-)holonomy in terms of (1-)torsor (1-)morphisms. The functor hol associates torsor morphisms between fibers to paths in the base manifold. Using trivializations t_i on patches U_i these torsor morphisms can be identified with elements of the structure group. The step $t_i^{-1} \circ t_j$ from one trivialization to another one on double overlaps U_{ij} gives rise to multiplication by the transition function g_{ij}.

12.4.1.2 Gauge Transformations. Changing from one local trivialization

$$i \mapsto \left(\mathrm{hol}|_{U_i} \xrightarrow{t_i} \mathrm{hol}_i \right)$$

to another one

$$i \mapsto \left(\mathrm{hol}|_{U_i} \xrightarrow{\tilde{t}_i} \tilde{\mathrm{hol}}_i \right)$$

is called a **gauge transformation**.

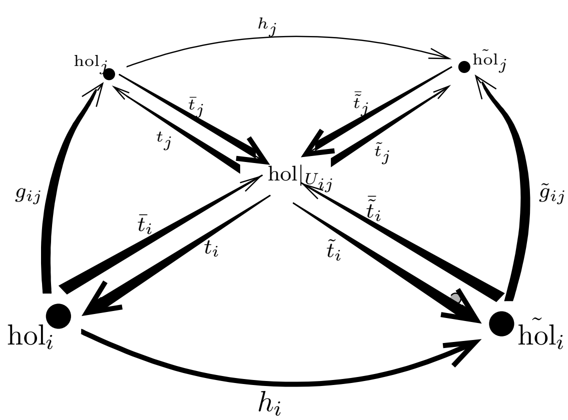
This corresponds to replacing the transition

$$\begin{tikzcd} & \mathrm{hol}|_{U_{ij}} \arrow[dr, "t_j"] & \\ \mathrm{hol}_i \arrow[ur, "\bar{t}_i"] \arrow[rr, "g_{ij}"'] & & \mathrm{hol}_j \end{tikzcd}$$

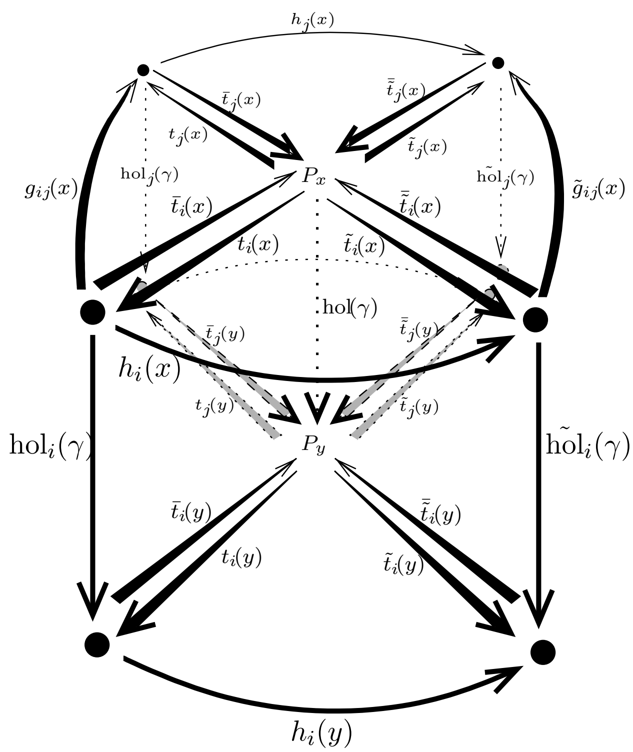
by another transition $\tilde{g}_{ij}$ which is given by the commutativity of this diagram:

$$\begin{tikzcd} & \mathrm{hol}|_{U_{ij}} \arrow[dr, "\tilde{t}_j"] & \\ \mathrm{hol}_i \arrow[ur, "\tilde{\bar{t}}_i"] \arrow[rr, "\tilde{g}_{ij}"'] & & \mathrm{hol}_j \end{tikzcd}$$

This situation is depicted by the following commuting diagram:



In components this means that the following diagram commutes, too:



From this diagram one reads off that

$$\tilde{g}_{ij} = h_i^{-1}\, g_{ij}\, h_j$$

and that

$$\tilde{\mathrm{hol}}_i = h_i^{-1}\, \mathrm{hol}_i\, h_i\,,$$

where

$$h_i = \bar{t}_i \circ \tilde{t}_i$$

and

$$h_j = \bar{t}_j \circ \tilde{t}_j\,.$$

This is the well-known gauge transformation law for 1-bundles with connection.

In fact, when read from the left to the right one sees that this diagram is equivalent to the diagram (12.1) (p. 289) which depicts a natural isomorphism of the global holonomy 1-functor regarded as a functor

$$\mathrm{hol}\colon \mathcal{P}_1^C(\mathcal{U}) \to G$$

from the Čech-extended 1-path 1-groupoid (def. 12.2, p. 286) of the good covering $\mathcal{U}$ to the structure group G. This justifies our use of the same symbol "hol" for both functors.

In the next subsection all this is categorified.

12.4.2 2-Torsors and 2-Bundles with 2-Holonomy

Let G_2 be any 2-group and let $G_2-2\mathrm{Tor}$ be the 2-category of G_2-2-torsors.

A principal G_2-2-bundle with connection and holonomy is specified by a 2-functor

$$\mathrm{hol}\colon \mathcal{P}_2(M) \to G_2-2\mathrm{Tor}\,.$$

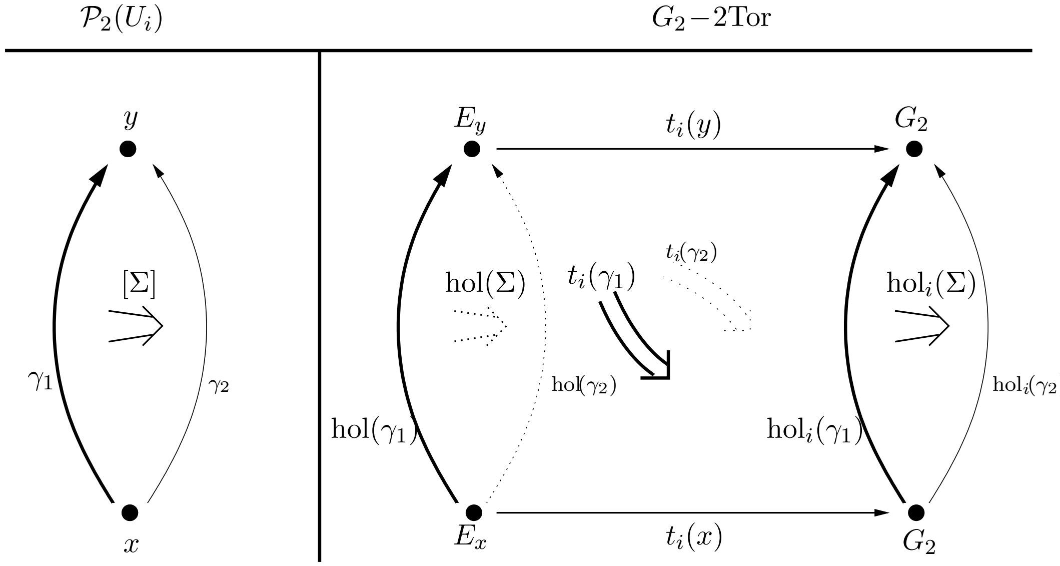
When restricted to any of the U_i, the 2-functor hol is pseudonaturally isomorphic to a local 2-holonomy 2-functor

$$\mathrm{hol}_i\colon \mathcal{P}_2(U_i) \to G_2\,.$$

Fix any such natural isomorphism

$$\mathrm{hol}|_{U_i} \xrightarrow{t_i} \mathrm{hol}_i\,.$$

It is specified by a naturality tincan diagram



(12.11)

Here we use the fact that G_2 itself is a G_2-torsor and that its automorphisms in $G_2-2\mathrm{Tor}$ correspond to (right)-multiplication with identity morphism in G_2 and that its 2-morphisms correspond to 2-torsor 2-morphisms between these.

It follows that on double overlaps we have pseudonatural isomorphisms

$$\mathrm{hol}_i \xrightarrow{g_{ij}} \mathrm{hol}_j$$

given by diagrams like this:

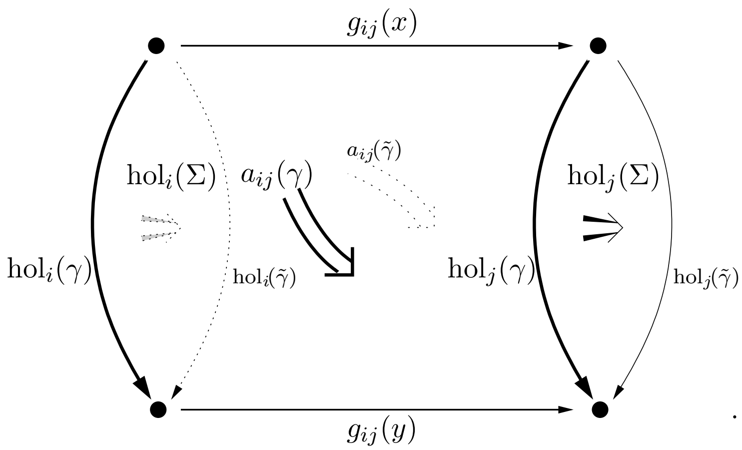
$$\begin{tikzcd}
& \mathrm{hol}|_{U_{ij}} \arrow[dr, "t_j"] & \\
\mathrm{hol}_i \arrow[ur, "\bar{t}_i"] \arrow[rr, "g_{ij}"'] \arrow[rr, phantom, shift left=4ex, "\Uparrow \phi_{ij}"] & & \mathrm{hol}_j
\end{tikzcd} \,. \tag{12.12}$$

Here $\bar{t}_i$ is any one weak inverse of t_i and

$$g_{ij} \xrightarrow{\phi_{ij}} \bar{t}_i \circ t_j$$

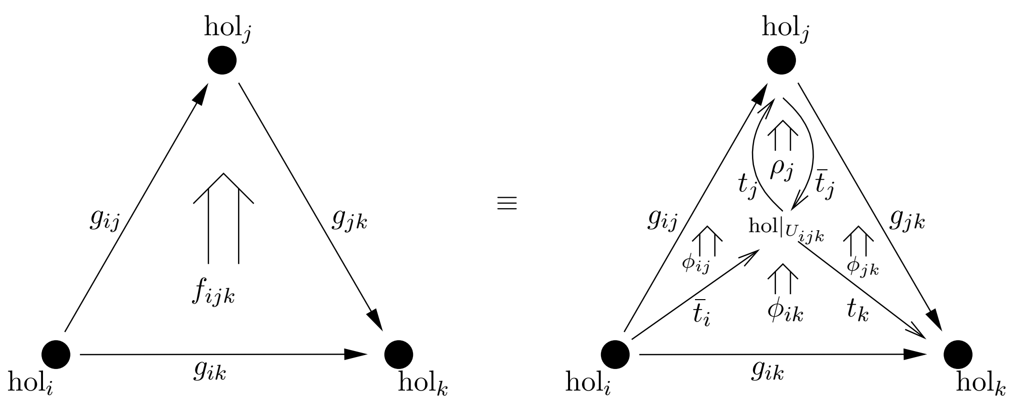
is a modification of pseudonatural transformations.

The pseudonatural transformation g_{ij} is given by a tincan diagram of this form:



. (12.13)

On triple overlaps g_{ik} and $g_{ij} \circ g_{jk}$ are related by a modification f_{ijk} which is a composite of three of the ϕ from (12.12) and of a $\rho_j : \mathrm{Id} \to t_j \circ \bar{t}_j$:



.

12.4.2.1 The various 2-morphisms involved. In the above diagrams we had, as usual, left all re-whiskering implicit. But for the following considerations it turns out that we need to take care of these details, lest a crucial point about the final result remains invisble.

Since we are working in the weak 2-category G_2–2Tor, there are several different things one could want to mean by “reversion” of a 2-morphism, depending on what we want to do to the source and target 1-mophisms. We shall define the following notation:

The 2-morphism in G_2-2Tor which we want to call t_i is precisely the following one:

$$
\begin{tikzcd}
G_2 \arrow[r, "\mathrm{hol}_i(\gamma)"] & G_2 \arrow[d, "\bar{t}_i(y)"] \\
E_x \arrow[u, "t_i(x)"] \arrow[r, "\mathrm{hol}|_{U_i}(\gamma)"'] \arrow[ur, phantom, "t_i(\gamma)\Uparrow"] & E_y
\end{tikzcd}
.
$$

Its inverse 2-morphism is this one:

$$
\begin{tikzcd}
G_2 \arrow[r, "\mathrm{hol}_i(\gamma)"] & G_2 \arrow[d, "\bar{t}_i(y)"] \\
E_x \arrow[u, "t_i(x)"] \arrow[r, "\mathrm{hol}|_{U_i}(\gamma)"'] \arrow[ur, phantom, "(t_i(\gamma))^{-1}\Downarrow"] & E_y
\end{tikzcd}
.
$$

We shall however be interested in the 2-morphism obtained from this one by reversing the 1-morphism on the left and the right. To that end we rewhisker, i.e. we horizontally compose $(t_i(\gamma))^{-1}$ with the identity 2-morphisms on $\bar{t}_i(x)$ and on $t_i(y)$:

$$
\begin{tikzcd}
& G_2 \arrow[r, "\mathrm{hol}_i(\gamma)"] & G_2 \arrow[d, "\bar{t}_i(y)"] & \\
& E_x \arrow[u, "t_i(x)"] \arrow[r, "\mathrm{hol}|_{U_i}(\gamma)"'] \arrow[ur, phantom, "(t_i(\gamma))^{-1}\Downarrow"] & E_y \arrow[dr, shift left, "t_i(y)"] \arrow[dr, shift right, "t_i(y)"'] & \\
G_2 \arrow[ur, shift left, "\bar{t}_i(x)"] \arrow[ur, shift right, "\bar{t}_i(x)"'] & & & G_2
\end{tikzcd}
.
$$

Since $\bar{t}_i$ is only the weak inverse of t_i, we furthermore need to compose with $\eta_i \equiv \mathrm{Id} \to \bar{t}_i \circ t_i$:

$$
\begin{tikzcd}
 & G_2 \arrow[r, "\mathrm{hol}_i(\gamma)"] & G_2 \arrow[d, "\bar{t}_i(y)"] \arrow[ddr, bend left, "\mathrm{Id}"] \arrow[ddr, phantom, "\Swarrow\,\eta_i(y)"] & \\
 & E_x \arrow[u, "t_i(x)"] \arrow[r, "\mathrm{hol}|_{U_i}(\gamma)"'] \arrow[ur, phantom, "(t_i(\gamma))^{-1}\Downarrow"] & E_y \arrow[dr, shift left, "t_i(y)"] \arrow[dr, shift right, "t_i(y)"'] & \\
G_2 \arrow[ur, shift left, "\bar{t}_i(x)"] \arrow[ur, shift right, "\bar{t}_i(x)"'] \arrow[uur, bend left, "\mathrm{Id}"] \arrow[uur, phantom, "\Searrow\,\eta_i(x)"] & & & G_2
\end{tikzcd}
.
$$

The resulting 2-morphism is the one we want to call $\bar{t}_i(\gamma)$:

$$
\begin{tikzcd}
G_2 \arrow[r, "\mathrm{hol}_i(\gamma)"] \arrow[d, "\bar{t}_i(x)"'] \arrow[dr, phantom, "\bar{t}_i(\gamma)\Downarrow"] & G_2 \\
E_x \arrow[r, "\mathrm{hol}|_{U_i}(\gamma)"'] & E_y \arrow[u, "t_i(y)"']
\end{tikzcd}
\equiv
\begin{tikzcd}
 & G_2 \arrow[r, "\mathrm{hol}_i(\gamma)"] & G_2 \arrow[d, "\bar{t}_i(y)"] \arrow[ddr, bend left, "\mathrm{Id}"] \arrow[ddr, phantom, "\Swarrow\,\eta_i(y)"] & \\
 & E_x \arrow[u, "t_i(x)"] \arrow[r, "\mathrm{hol}|_{U_i}(\gamma)"'] \arrow[ur, phantom, "(t_i(\gamma))^{-1}\Downarrow"] & E_y \arrow[dr, shift left, "t_i(y)"] \arrow[dr, shift right, "t_i(y)"'] & \\
G_2 \arrow[ur, shift left, "\bar{t}_i(x)"] \arrow[ur, shift right, "\bar{t}_i(x)"'] \arrow[uur, bend left, "\mathrm{Id}"] \arrow[uur, phantom, "\Searrow\,\eta_i(x)"] & & & G_2
\end{tikzcd}
. \qquad (12.14)
$$

Below, it turns out that we need to apply this process of going from $(t_i(\gamma))^{-1}$ to $\bar{t}_i(\gamma)$ the other way around. Since this is an important step, we go through it again:

So start this time with the above 2-morphism

$$
\begin{tikzcd}
E_x \arrow[r, "\mathrm{hol}|_{U_i}(\gamma)"] & E_y \arrow[d, "t_i(y)"] \\
G_2 \arrow[u, "\bar{t}_i(x)"] \arrow[r, "\mathrm{hol}_i(\gamma)"'] \arrow[ur, phantom, "\bar{t}_i(\gamma)\Uparrow"] & G_2
\end{tikzcd}
,
$$

and take its inverse

$$
\begin{tikzcd}
E_x \arrow[r, "\mathrm{hol}|_{U_i}(\gamma)"] \arrow[dr, phantom, "(\bar{t}_i(\gamma))^{-1}\Downarrow"] & E_y \arrow[d, "t_i(y)"] \\
G_2 \arrow[u, "\bar{t}_i(x)"] \arrow[r, "\mathrm{hol}_i(\gamma)"'] & G_2
\end{tikzcd},
$$

then re-whisker

$$
\begin{tikzcd}
& E_x \arrow[r, "\mathrm{hol}|_{U_i}(\gamma)"] \arrow[dr, phantom, "(\bar{t}_i(\gamma))^{-1}\Downarrow"] & E_y \arrow[d, "t_i(y)"] & \\
& G_2 \arrow[u, "\bar{t}_i(x)"] \arrow[r, "\mathrm{hol}_i(\gamma)"'] & G_2 \arrow[dr, shift left, "\bar{t}_i(y)"] \arrow[dr, shift right, "\bar{t}_i(y)"'] & \\
E_x \arrow[ur, shift left, "t_i(x)"] \arrow[ur, shift right, "t_i(x)"'] & & & E_x
\end{tikzcd},
$$

and compose with $\rho_i \equiv \mathrm{Id} \to t_i \circ \bar{t}_i$

$$
\begin{tikzcd}
& E_x \arrow[r, "\mathrm{hol}|_{U_i}(\gamma)"] \arrow[dr, phantom, "(\bar{t}_i(\gamma))^{-1}\Downarrow"] & E_y \arrow[d, "t_i(y)"] \arrow[ddr, bend left, "\mathrm{Id}"] \arrow[dr, phantom, "\Swarrow \rho_i(y)"] & \\
\arrow[ur, phantom, "\Searrow \rho_i(x)"] & G_2 \arrow[u, "\bar{t}_i(x)"] \arrow[r, "\mathrm{hol}_i(\gamma)"'] & G_2 \arrow[dr, shift left, "\bar{t}_i(y)"] \arrow[dr, shift right, "\bar{t}_i(y)"'] & \\
E_x \arrow[ur, shift left, "t_i(x)"] \arrow[ur, shift right, "t_i(x)"'] \arrow[uur, bend left, "\mathrm{Id}"] & & & E_y
\end{tikzcd}.
$$

The resulting 2-morphism must be $t_i(\gamma)$:

$$
\begin{tikzcd}
E_x \arrow[r, "\mathrm{hol}|_{U_i}(\gamma)"] \arrow[d, "t_i(x)"'] \arrow[dr, phantom, "t_i(\gamma)\Downarrow"] & E_y \\
G_2 \arrow[r, "\mathrm{hol}_i(\gamma)"'] & G_2 \arrow[u, "\bar{t}_i(y)"']
\end{tikzcd}
=
\begin{tikzcd}
& E_x \arrow[r, "\mathrm{hol}|_{U_i}(\gamma)"] \arrow[dr, phantom, "(\bar{t}_i(\gamma))^{-1}\Downarrow"] & E_y \arrow[d, "t_i(y)"] \arrow[dr, bend left, "\mathrm{Id}"] & \\
& G_2 \arrow[u, "\bar{t}_i(x)"] \arrow[r, "\mathrm{hol}_i(\gamma)"'] & G_2 \arrow[dr, shift left, "\bar{t}_i(y)"] \arrow[dr, shift right, "\bar{t}_i(y)"'] \arrow[r, phantom, "\rho_i(y)"] & E_y \\
E_x \arrow[ur, shift left, "t_i(x)"] \arrow[ur, shift right, "t_i(x)"'] \arrow[uur, bend left, "\mathrm{Id}"] \arrow[r, phantom, "\rho_i(x)"] & & &
\end{tikzcd}
\quad . \qquad (12.15)
$$

We can define the a_{ij} as the composition of $\bar{t}_i(\gamma)$ with $t_j(\gamma)$ up to modification ϕ_{ij}

$$
a_{ij}(\gamma) \xrightarrow{\phi_{ij}} \bar{t}_i(\gamma) \circ t_j(\gamma)
$$

as the composition $\bar{t}_i(\gamma) \circ t_j(\gamma)$ with boundary

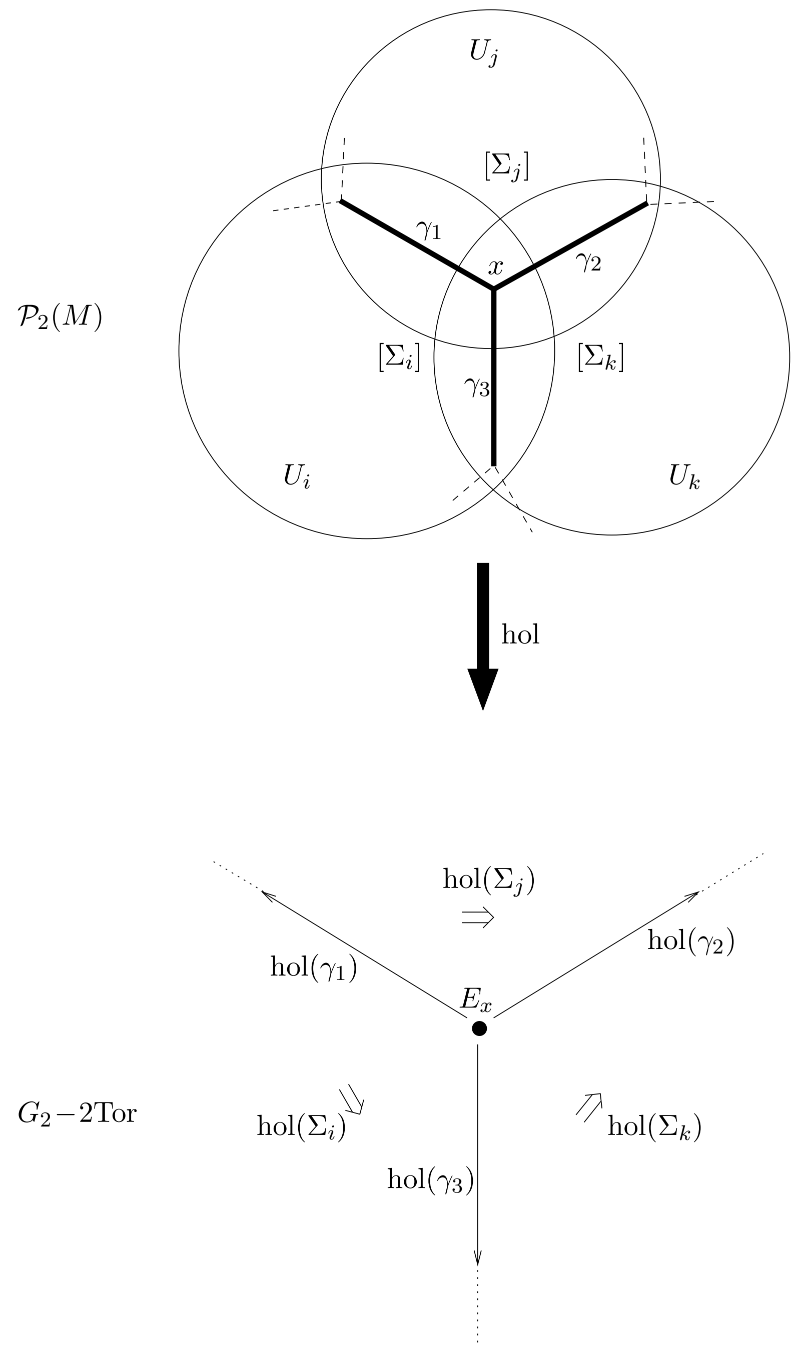
$$
g_{ij}(x) \xrightarrow{\phi_{ij}(x)} \bar{t}_i(x) \circ t_j(x) \ .
$$

In order to perform this compositon we need to rewhisker t_j from the left by the identity 2-morphism $\mathrm{Id}_{\bar{t}_i}$ and from the right by the identity 2-morphism Id_{t_i}:

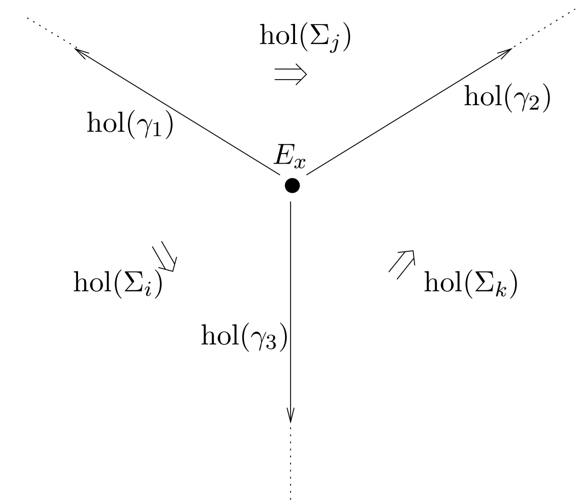
$$
\begin{tikzcd}
& G_2 \arrow[r, "\mathrm{hol}_j(\gamma)"] & G_2 \arrow[dr, "\bar{t}_j(y)"] \arrow[dd, "g_{ij}^{-1}(y)"'] & \\
\bullet \arrow[ur, "t_j(x)"] \arrow[r, phantom, "\Uparrow \phi_{ij}(x)"] & \arrow[r, phantom, "\Uparrow a_{ij}(\gamma)"] & \arrow[r, phantom, "\Uparrow \phi_{ij}(y)"] & \bullet \arrow[dl, "t_i(y)"] \\
& G_2 \arrow[ul, "\bar{t}_i(x)"] \arrow[uu, "g_{ij}(x)"] \arrow[r, "\mathrm{hol}_i(\gamma)"'] & G_2 &
\end{tikzcd}
\equiv
\begin{tikzcd}
G_2 \arrow[r, "\mathrm{hol}_j(\gamma)"] \arrow[dr, phantom, "\Uparrow t_j(\gamma)"] & G_2 \arrow[d, "\bar{t}_j(y)"] \\
E_x \arrow[u, "t_j(x)"] \arrow[r, "\mathrm{hol}|_{U_{ij}}(\gamma)"] \arrow[dr, phantom, "\Uparrow \bar{t}_i(\gamma)"] & E_y \arrow[d, "t_i(y)"] \\
G_2 \arrow[u, "\bar{t}_i(x)"] \arrow[r, "\mathrm{hol}_i(\gamma)"'] & G_2
\end{tikzcd}
\equiv
\begin{tikzcd}
G_2 \arrow[rrr, "\mathrm{hol}_j(\gamma)"] \arrow[drrr, phantom, "\Uparrow t_j(\gamma)"] & & & G_2 \arrow[d, "\bar{t}_j(y)"] \\
E_x \arrow[u, "t_j(x)"] \arrow[rrr, "\mathrm{hol}|_{U_{ij}}(\gamma)"] & & & E_y \arrow[dd, shift left, "t_i(y)"] \arrow[dd, shift right] \\
& \bullet \arrow[r, "\mathrm{hol}|_{U_{ij}}(\gamma)"] \arrow[dr, phantom, "\Uparrow \bar{t}_i(\gamma)"] & \bullet \arrow[d, "t_i(y)"] & \\
\bullet \arrow[uu, shift left] \arrow[uu, shift right, "\bar{t}_i(x)"] & G_2 \arrow[u, "\bar{t}_i(x)"] \arrow[r, "\mathrm{hol}_i(\gamma)"'] & G_2 & \bullet
\end{tikzcd}
$$

(12.16)

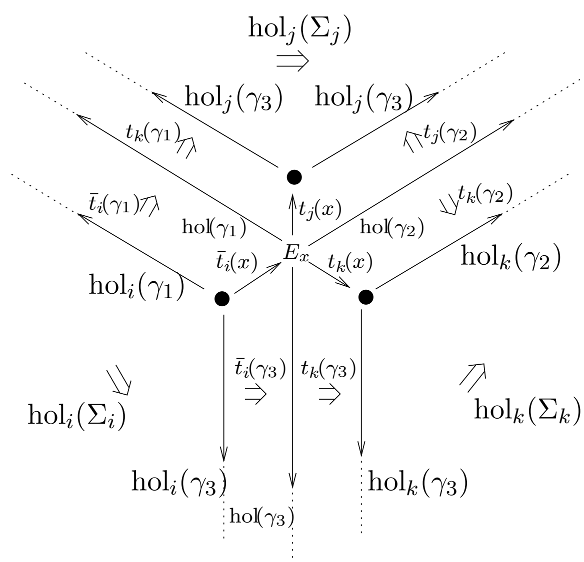
12.4.2.2 Surface Holonomy in Terms of Local Trivializations. Now consider the application of hol to any 2-morphism $[\Sigma] \in \mathcal{P}_2(M)$ that does not necessarily sit in a single patch.



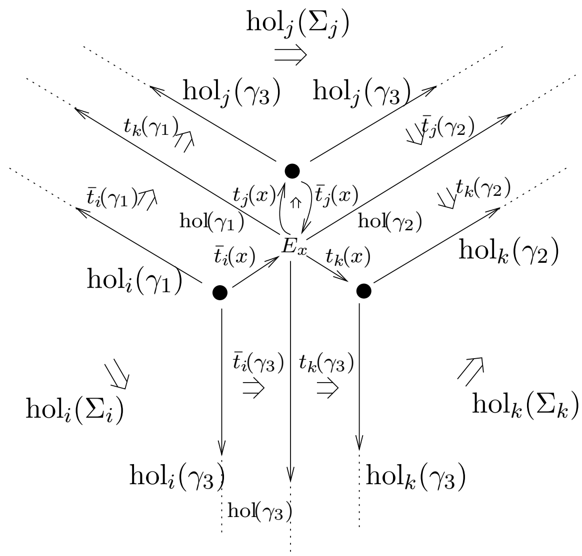
We can decompose $[\Sigma]$ into several 2-morphisms that all do sit inside a single patch of the covering. Their images under hol can be regarded as the "bottom" (leftmost surface) of the tincan diagram (12.11). Since this tincan diagram 2-commutes, we can replace each $\mathrm{hol}(\Sigma_i)$ by the the respective tincan with its leftmost surface cut out. This is shown in the following figure.



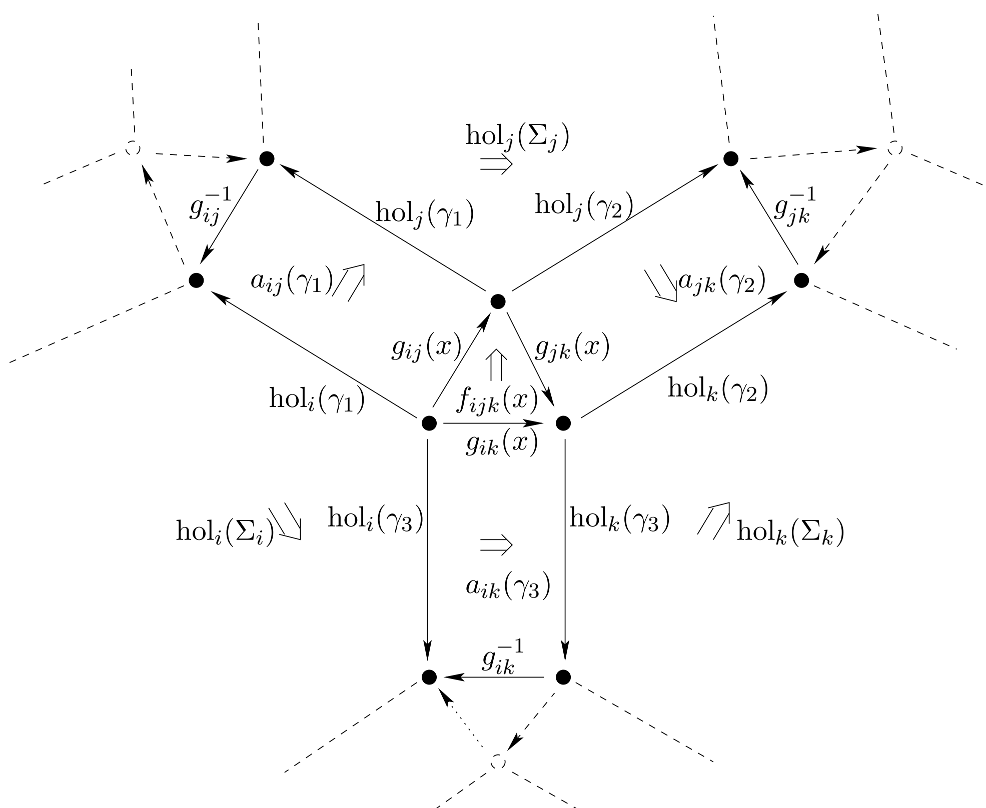
$$=$$



We can express $t_j(\gamma_2)$ in terms of $\bar{t}_j(\gamma_2)$, using equation (12.15). This introduces the 2-morphism $\mathrm{Id} \xrightarrow{\rho_j} t_j \circ \bar{t}_j$ into the diagram:



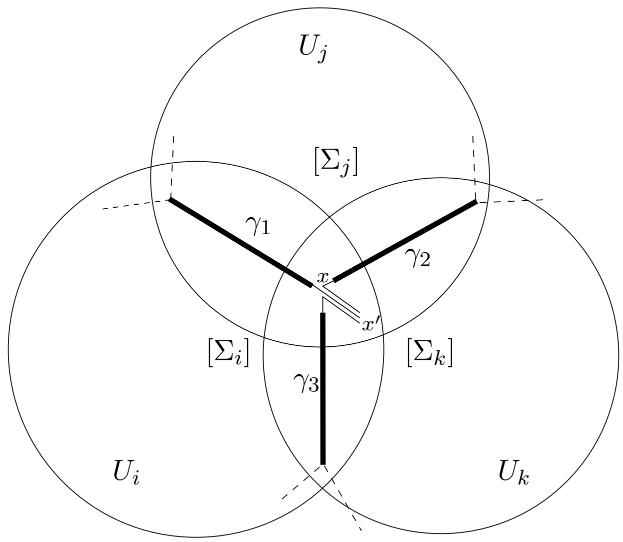
Then we can compose the t and $\bar t$ pairwise, using (12.16). Compare this to the discussion in §11.1 (p.242). The result is the diagram already familiar from §12.2.2 (p.293):



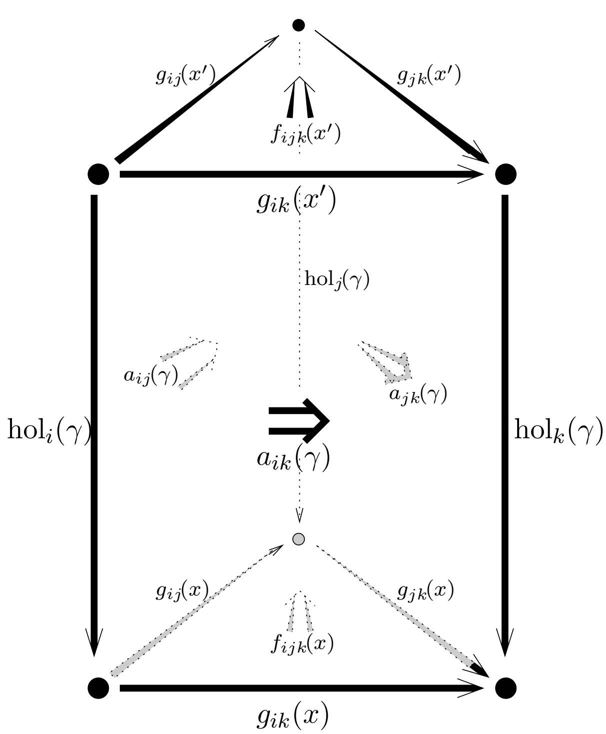
This shows how hol: $\mathcal{P}_2(M) \to G_2$–2Tor can be reexpressed in terms of hol$_i$: $\mathcal{P}_2(U_i) \to G$ on each U_i.

12.4.2.3 Consistency conditions. There are two consistency conditions on the t_i.

One comes from the condition that degenerate surfaces do not contribute. Consider moving $x \in U_{ijk}$ to $x' \in U_{ijk}$ by extending all of γ_1, γ_2, γ_3 by the *same* path $x \to x'$.

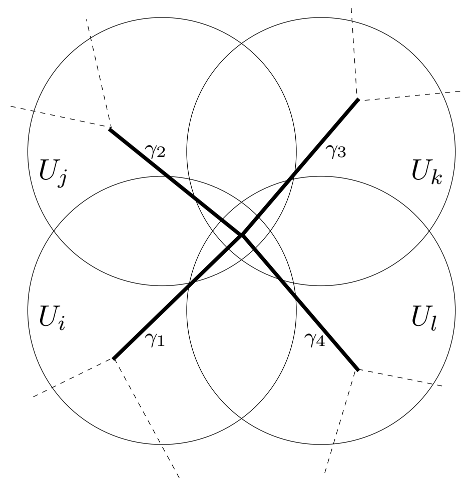


This removes $f_{ijk}(x)$ in the above diagram and replaces it by a diagram of the above form around x' but with all Σ_i vanishing. Since this must not contribute, this diagram has to equal the 2-morphism $f_{ijk}(x)$ that was replaced. In other words, the a_{ij} must be such that the following diagram 2-commutes:



This is the transition law on triple overlaps discussed in §11.4.3 (p.266).

The other consistency condition is obtained by considering vertices at which more than three edges meet. Whenever this is the case, we can insert constant paths until only trivalent vertices are left. But these constant paths can be inserted in more than one way. Four a 4-valent vertex this is indicated by the following figure.

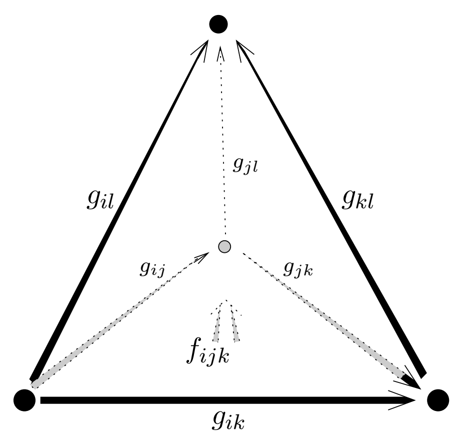


Here γ_0 denotes a constant path, which has been drawn with a spatial extension just for convenience. It is a well known theorem (see for instance [84]) that all triangulations can be obtained from any given one by a series of moves of two types, one of which is the one from one of the lower two pictures to the other. The other is the "bubble move" which was called the "left and right unit law" in §11.2.4 (p.251).

Invariance of hol under this move is expressed by the equation

$$
\begin{tikzcd}
\mathrm{hol}_j \arrow[r, "g_{jk}"] & \mathrm{hol}_k \arrow[d, "g_{kl}"] \\
\mathrm{hol}_i \arrow[u, "g_{ij}"] \arrow[ur, "g_{ik}"] \arrow[r, "g_{il}"'] \arrow[ur, phantom, shift left=4, "\Uparrow f_{ijk}"] \arrow[ur, phantom, shift right=4, "\Uparrow f_{ikl}"'] & \mathrm{hol}_l
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathrm{hol}_j \arrow[r, "g_{jk}"] \arrow[dr, "g_{jl}"'] \arrow[dr, phantom, shift left=4, "\Uparrow f_{jkl}"] & \mathrm{hol}_k \arrow[d, "g_{kl}"] \\
\mathrm{hol}_i \arrow[u, "g_{ij}"] \arrow[r, "g_{il}"'] \arrow[ur, phantom, shift right=4, "\Uparrow f_{ijl}"'] & \mathrm{hol}_l
\end{tikzcd},
$$

which is equivalent to the 2-commutativity of this tetrahedron:



This is the tetrahedron transition law on quadruple overlaps known from equation (11.15) (p. 260) and from §11.2.4 (p.251).

12.4.2.4 Gauge Transformations. The discussion of gauge transformations completely parallels that in §12.4.1.2 (p.319), only that now nontrivial 2-morphisms appear where previously only identity 2-morphisms were present.

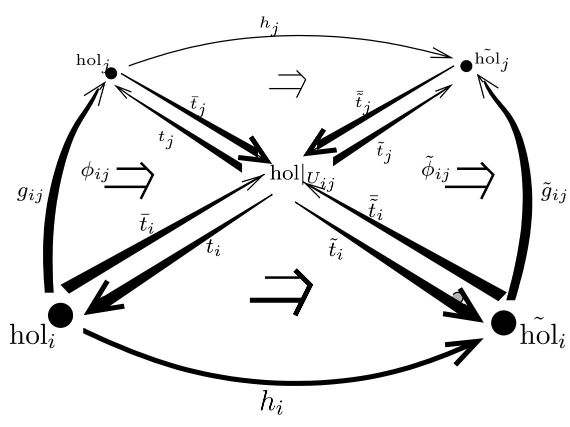
Switching from one local trivialization to another corresponds to replacing the transition

$$
\begin{tikzcd}
& \mathrm{hol}|_{U_{ij}} \arrow[dr, "t_j"] & \\
\mathrm{hol}_i \arrow[ur, "\bar{t}_i"] \arrow[rr, "g_{ij}"'] \arrow[rr, phantom, shift left=5, "\Uparrow \phi_{ij}"] & & \mathrm{hol}_j
\end{tikzcd}
\quad . \tag{12.17}
$$

by another transition

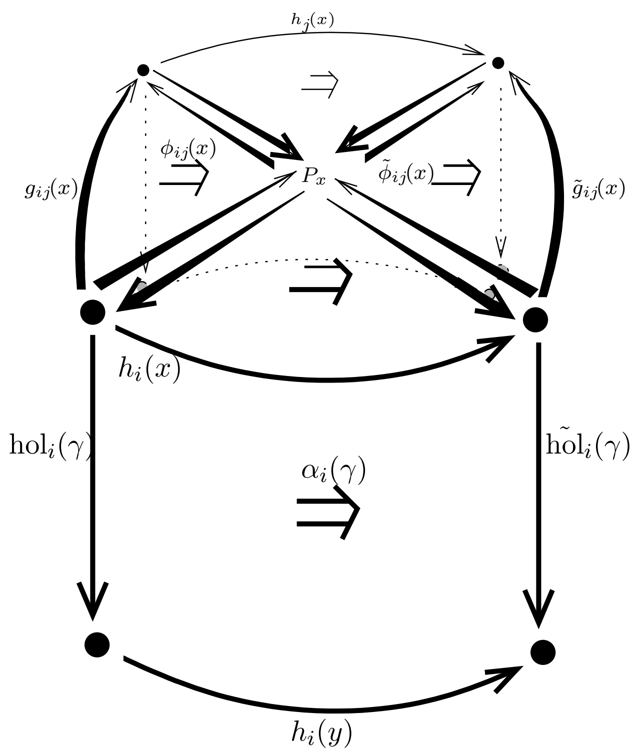
$$
\begin{tikzcd}
 & \mathrm{hol}|_{U_{ij}} \arrow[dr, "\tilde{t}_j"] & \\
\tilde{\mathrm{hol}}_i \arrow[ur, "\bar{\tilde{t}}_i"] \arrow[rr, "\tilde{g}_{ij}"'] \arrow[rr, phantom, bend left=40, "\Uparrow \tilde{\phi}_{ij}"] & & \tilde{\mathrm{hol}}_j
\end{tikzcd}
\quad . \tag{12.18}
$$

Hence we get the following diagram



The existence of these modifications of pseudonatural transformations implies that a

diagram as indicated in the following figure 2-commutes:



(For readability, not all 2-morphism are shown.) But this is nothing but the naturality tincan diagram describing the gauge transformations discussed in §12.2.3 (p.297) This means that our 2-holonomy 2-functor

$$\mathrm{hol}\colon \mathcal{P}_2(M) \to \mathrm{Trans}_2(E)$$

encodes the same information as gauge equivalence classes of 2-functors

$$\mathrm{hol}\colon \mathcal{P}_2^C(\mathcal{U}) \to G_2$$

from the Čech-extended 2-path 2-groupoid to the structure 2-group, defined in §12.2 (p.291).

13. The Differential Picture: Nonabelian Deligne Hypercohomology

We now study the "differential version" of the considerations in §11 (p.242) and §12 (p.285).

The differential version of a (Lie-)p-groupoid is called a (Lie-)p-algebroid [40]. The differential version of a holonomy functor between p-groupoids should be a morphisms between p-algebroids. Similarly, the differential version of a natural transformation between two such p-functors should be a 2-isomorphism between morphisms of p-algebroids, and so on. This is indicated on the right of figures 6, p. 19 and 7, p. 21.

It is known that Lie p-algebras (and Lie p-algebroids) are equivalently described in terms of L_∞ algebras [40] which again are known to be described dually in terms of differential graded algebras (dg-algebras) [194, 185, 195]. Explicit translations from p-algebroids to their dual dg-algebras are, for $p = 1$ and $p = 2$, spelled out in [196, 197].

In the following we first review aspects of the formalism of expresssing p-algebroid morphisms in terms of dg-algebra morphisms, following [196, 197, 198]. Then we discuss how the differential picture of what was done in §12 (p.285) has an analogue in this formalism, thereby finding a generalized form of Deligne hypercohomology.

Finally we check for the special case of (strict) nonabelian 1- and 2-bundles that the cocycle relations generated by nonabelian Deligne cohomology do indeed reproduce the infinitesimal version of the full cocycle relations known from the integral picture.

13.1 Introduction

Abelian gerbes with connection and curving are well known to be given by classes in Deligne hypercohomology [178, 177, 47, 48], which is the combination of Čech and de Rham cohomology.

More recently, nonabelian (bundle-)gerbes have been studied in more detail [49, 50], but without an appropriate nonabelian generalization of abelian Deligne cohomology available.

But abelian and nonabelian gerbes may equivalently be described in terms of categorified principal fiber bundles [36, 31]. These can be described in terms of p-functors hol_i from certain p-groupoids $\mathcal{P}_p(U_i)$ of paths to the structure p-group(oid) G_p. (Here $p = 1$ gives an ordinary bundle and $p = 2$ a 2-bundle, or gerbe.)

Similar to how a Lie goup has a differential description in terms of its Lie algebra, there should be a differential description of these p-functors in terms of morphisms between p-algebroids.

Such morphisms have been studied in detail in the context of certain topological σ-models [196, 197], and aspects of their relation to gerbes have been indicated in [37, 199]. A powerful tool used in these studies is the description of p-algebroids in terms of their *duals*, which are nothing but differential graded algebras (dg-algebras). In this language a morphism of p-algebroids corresponds to a chain map between dg-algebras, a 2-morphisms between two such morphisms corresponds to a chain homotopy, and so on. There is a natural operator, Q, which makes the space of dg-algebra n-morphisms into a complex.

We will discuss how the differential version of the description of p-bundles with p-connection in terms of p-functors from a p-groupoid of p-paths to the structure p-group(oid) corresponds to an assignment of dg-algebra $(n+1)$-morphisms to Čech-n-simplices. In

particular, the path groupoid $\mathcal{P}_p(U_i)$ becomes a dg-algebra dual to an algebroid $\mathfrak{p}_p(U_i)$ and the structure p-group(oid) G_p becomes a dg-algebra dual to the structure p-algebra or structure p-algebroid $\mathfrak{g}_p$. Moreover, any such assignment corresponds to an element in the kernel of a generalized noncommutative Deligne coboundary operator, which is obtained from the ordinary Deligne operator, by, roughly, substituting the operator Q for the de Rham differential.

We demonstrate this explicitly for ordinary principal bundles, as well as for principal 2-bundles with strict structure 2-group (which correspond to nonabelian gerbes), checking that the generalized Deligne closed-ness condition reproduces the "infinitesimal" version of the known cocycle conditions and gauge transformation laws for these structures.

However, the approach presented here applies without any changes to much more general situations than these. By choosing appropriate target dg-algebras one obtains the infinitesimal version of p-bundles with p-connection whose "structure group" may be semistrict instead of strict, or even be a groupoid instead of a group. In fact, there are semistrict Lie p-algebras which cannot be integrated to any Lie p-group, but which can nevertheless be used in the formalism of generalized Deligne cohomology.

On the other hand, this points to a general issue that we do not try to address here, namely the question concerning what one would like to address as the process of "integrating" a class in generalized Deligne cohomology to a proper p-bundle with p-connection. As long as this question is open in the general case, we cannot say how the cohomology classes of the generalized Deligne operator correspond exactly to gauge equivalence classes of locally trivialized p-bundles with p-connection.

13.2 Preliminaries

There are some standard concepts and definitions that we should recall in order to fix notation and nomenclature:

13.2.1 Čech-Simplices

Consider a given manifold M called the **base manifold** together with a **good covering**

$$\mathcal{U} \to M\,.$$

So for some countable index set I, $\mathcal{U}$ is a collection

$$\mathcal{U} = \bigsqcup_{i\in I} U_i$$

of open subsets $U_i \subset M$, such that M is covered by these,

$$M = \bigcup_{i\in I} U_i\,,$$

and such that every nonempty finite intersection

$$U_{i_1 i_2 \dots i_n} \equiv U_{i_1} \cap U_{i_2} \cap \dots \cap U_{i_n}\,, \qquad \forall\, n = 1, 2, \dots$$

is contractible.

Locally trivializing a (categorified) bundle with respect to $\mathcal{U}$ involves specifying transition functions and transition functions between transition functions associated to "Čech-simplices". These simplices are elements of the Čech complex:

Definition 13.1 *The pth* **Čech chain-complex**

$$C(\mathcal{U}) = \bigoplus_{n=0}^{p} C_n(\mathcal{U})$$

is the free abelian group generated by all tuples $(i_0 i_1 \ldots i_n) \in I^{n+1}$, for $n \leq p$, together with the linear nilpotent **Čech boundary operator**

$$\begin{aligned} \delta|_{C_n} \equiv \delta_n : \quad & C_n(\mathcal{U}) && \to C_{n-1}(\mathcal{U}) \\ & (i_0 i_1 \cdots i_n) && \mapsto -\sum_{m=0}^{n} (-1)^m (i_0 i_1 \cdots \widehat{i_m} \cdots i_n)\,, \end{aligned}$$

where, as usual, $\widehat{i_m}$ indicates that the element i_m is to be omitted.

Hence we have a chain complex

$$0 \xrightarrow{0} C_p \xrightarrow{\delta} C_{p-1} \xrightarrow{\delta} \ldots C_0 \xrightarrow{\delta} 0\,.$$

The tuples $(i_0 \ldots i_n)$ are called **Čech-n-simplices** and arbitrary linear combinations of them (over $\mathbb{Z}$) are called **Čech-chains**.

Definition 13.2 *We shall write $C^{\pm}(\mathcal{U}) \subset C(\mathcal{U})$ for the subset of elements of positive or negative coeffient, respectively, and*

$$P_{\pm} : C(\mathcal{U}) \to C^{\pm}(\mathcal{U})$$

for the obvious projection operation. For any given chain $c \in C(\mathcal{U})$ the chain δc is of course called the **boundary** *of c and the elements*

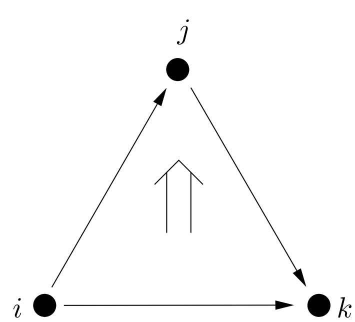
$$\begin{aligned} s(c) &\equiv P_+(\delta c) \\ t(c) &\equiv -P_-(\delta c) \end{aligned}$$

are called the **source boundary** *and the* **target boundary** *of c, respectively.*

We have

$$\delta c = s(c) - t(c)\ .$$

For example (ijk) describes the triangle



and $\delta(ijk) = (ik) - (ij) - (jk)$ are the arrows making up its boundary, with $s(ijk) = (ik)$ being the source boundary and $t(ijk) = (ij) + (jk)$ being the target boundary.

13.2.2 Differential Graded Algebras

Definition 13.3 *A* **differential graded algebra** *or* **dg-algebra** $(\mathbf{d}^V, \bigwedge^\bullet V^*)$ *is a graded vector space*

$$V^* = \bigoplus_n V_n^*$$

over some field k, with a graded commutative algebra product

$$\wedge \colon V^* \times V^* \to V^* ,$$

together with a k-linear operator

$$\mathbf{d}^V : \bigwedge^n V^* \to \bigwedge^{(n+1)} V^*$$

of degree +1 that is nilpotent

$$(\mathbf{d}^V)^2 = 0$$

and that satisfies the graded Leibniz rule

$$\mathbf{d}^V(\alpha \wedge \beta) = (\mathbf{d}^V \alpha) \wedge \beta + (-1)^{|\alpha|} \alpha \wedge (\mathbf{d}^V \beta) ,$$

for all $\alpha \in \bigwedge^{|\alpha|} V^*$ *and* $\beta \in \bigwedge^{|\beta|} V^*$.

Note that

$$\begin{aligned}
\bigwedge^1 V^* &= V_1^* \\
\bigwedge^2 V^* &= (V_1^* \bigwedge V_1^*) \oplus V_2^* \\
\bigwedge^3 V^* &= (V_1^* \bigwedge V_1^* \bigwedge V_1^*) \oplus (V_1^* \bigwedge V_2^*) \oplus V_3^* \\
&\vdots
\end{aligned}$$

The maximal grade of the graded vector space V^* of a dg-algebra corresponds to the maximal dimension of nontrivial morphisms in the dual L_∞-algebra. Therefore

Definition 13.4 *We shall call a dg-algebra* $(\mathbf{d}^V, \bigwedge^\bullet V^*)$ **of level** p *if*

$$V_n^* = 0, \forall\, n < 0,\ n > p .$$

The nice thing about the dg-algebra description of p-algebroids and L_∞-algebras is that the notion of morphism between dg-algebras is very convenient (*cf.* prop. 2 in [196]). It is nothing but a chain map:

Definition 13.5 *A* **morphism between dg-algebras**

$$(\mathbf{d}^A, \overset{\bullet}{\bigwedge} A^*) \xrightarrow{f} (\mathbf{d}^B, \overset{\bullet}{\bigwedge} B^*)$$

is a chain map*, i.e. a linear, grade preserving map*

$$f\colon \overset{\bullet}{\bigwedge} B^* \to \overset{\bullet}{\bigwedge} A^* ,$$

such that all these diagrams commute:

$$\begin{tikzcd}
\cdots \arrow[r, dotted] & \bigwedge^1 B^* \arrow[r, "\mathbf{d}^B"] \arrow[d, "f_1"] & \bigwedge^2 B^* \arrow[r, "\mathbf{d}^B"] \arrow[d, "f_2"] & \bigwedge^3 B^* \arrow[r, "\mathbf{d}^B"] \arrow[d, "f_3"] & \bigwedge^4 B^* \arrow[r, dotted] \arrow[d, "f_4"] & \cdots \\
\cdots \arrow[r, dotted] & \bigwedge^1 A^* \arrow[r, "\mathbf{d}^A"'] & \bigwedge^2 A^* \arrow[r, "\mathbf{d}^A"'] & \bigwedge^3 A^* \arrow[r, "\mathbf{d}^A"'] & \bigwedge^4 A^* \arrow[r, dotted] & \cdots
\end{tikzcd}$$

This means that

$$f_n \circ \mathbf{d}^A - \mathbf{d}^B \circ f_{n+1} = 0 , \qquad \forall\, n \in \mathbb{Z} .$$

For handling such morphisms it is very convenient (and in fact crucial for the definition of generalized Deligne cohomology in §13.4 (p.350)) to consider the direct sum complex:

Definition 13.6 *Given complexes* $(\mathbf{d}^A, \bigwedge^\bullet A^*)$ *and* $(\mathbf{d}^B, \bigwedge^\bullet B^*)$*, their* **direct sum complex** *is*

$$\left(Q\,, \overset{\bullet}{\bigwedge} A^* \oplus \overset{\bullet}{\bigwedge} B^*\right)$$

with differential

$$Q \equiv \mathbf{d}^A \oplus \mathbf{d}^B = \begin{bmatrix} \mathbf{d}^A & 0 \\ 0 & \mathbf{d}^B \end{bmatrix} .$$

We may naturally identifiy every map $\bigwedge^\bullet B^* \xrightarrow{f} \bigwedge^\bullet A^*$ with a map $\begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix}$ of the direct sum complex to itself.

Using this, the above condition for the chain map f simply says that **a chain map is Q-closed**:

$$[Q, f] = 0 .$$

A chain map is quite general a concept. We shall be interested frequently in certain special cases of chain maps. For instance in those with the following property:

Definition 13.7 *We shall call a morphism f of dg-algebras*

$$(\mathbf{d}^B, \bigwedge^{\bullet} B^*) \xrightarrow{f} (\mathbf{d}^A, \bigwedge^{\bullet} A^*)$$

homogenizing *iff*

$$f\left(\bigwedge^{n} B^*\right) \subset A^*_n \subset \bigwedge^{n} A^* \,.$$

Since 1-morphisms of dg-algebras are nothing but chain maps, 2-morphisms of dg-algebras are nothing but chain homotopies:

Definition 13.8 *A* **2-morphisms between dg-algebra morphisms**

$$\epsilon \colon f \to g$$

with

$$f, g \colon (\mathbf{d}^B, \bigwedge^{\bullet} B^*) \to (\mathbf{d}^A, \bigwedge^{\bullet} A^*)$$

is a **chain homotopy**, *i.e. a linear map*

$$\epsilon \colon \bigwedge^{\bullet} B^* \to \bigwedge^{(\bullet - 1)} A^*$$

such that

$$f_n - g_n = \mathbf{d}^B \circ \epsilon_{n+1} + \epsilon_n \circ \mathbf{d}^A \,.$$

The following diagram illustrates this situation (but note that the triangles in this diagram are *not* supposed to commute):

$$\begin{tikzcd}
{} \arrow[r, dotted] & \bigwedge^1 B^* \arrow[r, "\mathbf{d}^B"] \arrow[d, "f_1 - g_1"] \arrow[dl, dotted, dash] & \bigwedge^2 B^* \arrow[r, "\mathbf{d}^B"] \arrow[d, "f_2 - g_2"] \arrow[dl, "\epsilon^1"] & \bigwedge^3 B^* \arrow[r, "\mathbf{d}^B"] \arrow[d, "f_3 - g_3"] \arrow[dl, "\epsilon^2"] & \bigwedge^4 B^* \arrow[r, dotted] \arrow[d, "f_4 - g_4"] \arrow[dl, "\epsilon^3"] & {} \arrow[dl, dotted] \\
{} \arrow[r, dotted] & \bigwedge^1 A^* \arrow[r, "\mathbf{d}^A"] & \bigwedge^2 A^* \arrow[r, "\mathbf{d}^A"] & \bigwedge^3 A^* \arrow[r, "\mathbf{d}^A"] & \bigwedge^4 A^* \arrow[r, dotted] & {}
\end{tikzcd}$$

Using the language of the direct sum complex (def. 13.6), this may be expressed by saying that **a chain homotopy is a shift by a Q-exact term**:

$$f = g + [Q, \epsilon] \,.$$

(Note that $[,]$ denotes the graded commutator and that ϵ here is of odd degree.)

The formulation in terms of Q immediately suggests how to define dg-algebra n-morphisms for arbitrary n:

Definition 13.9 *Given dg-algebras* $(\mathbf{d}^A, \bigwedge^\bullet A^*)$ *and* $(\mathbf{d}^B, \bigwedge^\bullet B^*)$*, we say that for* $n \geq 2$ *an* n**-morphism between dg-algebra** $(n-1)$ **morphisms**

$$\epsilon\colon \phi \to \gamma$$

with

$$\phi, \gamma\colon \bigwedge^{\bullet} B^* \to \bigwedge^{\bullet - n + 2} A^*$$

is a linear map

$$\epsilon\colon \bigwedge^{\bullet} B^* \to \bigwedge^{\bullet - n + 1} A^*$$

such that ϕ *and* γ *differ by the* Q*-exact term* $[Q, \epsilon]$*:*

$$\phi - \gamma = [Q, \epsilon]\,.$$

This simple definition of maps between dg-algebras in terms of Q**-cohomology** turns out to capture all there is to say about n-morphisms between dg-algebras and hence about n-morphisms between p-algebroids.

So we now turn to the definition of the dg-algebra morphism to be called a local p-connection and the n-morphisms related to that.

13.3 The p-Connection Morphism

We have discussed in §12 (for $p = 1,2$) that a p-bundle with p-connection locally gives rise to functors

$$\mathcal{P}_p(U_i) \xrightarrow{\mathrm{hol}_i} G_p$$

from the p-path p-groupoid $\mathcal{P}_p(U_i)$ of a given patch U_i to the structure p-group(oid) G_p. We would like to find the analogous differential version of these functors.

Differentiating the source and target p-group(oid)s gives rise to source and target p-algebras (p-algebroids). The hol_i functors hence should become morphisms of p-algebroids, or, dually, morphisms of the associated dg-algebras (def. 13.5).

We shall, however, *not* try here to give a precise definition of this differentiation procedure of p-functors between p-groupoids. Instead, we will contend ourselves with proposing a certain obvious class of dg-algebra morphisms and demonstrating that the nonabelian Deligne cohomology obtained from them does have the right properties to be the infinitesimal description of the integral picture described in §12.

To that end, we discuss in the following first the target dg-algebra $\mathfrak{g}_p$ that is to replace the target p-group(oid) G_p, then the source dg-algebra $\mathfrak{p}_p(U_i)$ that is to replace the source p-path p-groupoid $\mathcal{P}_p$ and finally the morphisms of dg-algebras

$$\mathfrak{p}_p(U_i) \xrightarrow{\mathrm{con}_i} \mathfrak{g}_p$$

that is to replace the p-holonomy p-functor between these.

13.3.1 The Target dg-Algebra

For demonstrating the consistency of the differential picture with the integral picture, we shall be interested in the special case where the structure p-group G_p is a strict p-group, for $p = 1,2$.

For $p = 1$ the differential version of a strict p-group is simply the ordinary Lie algebra $\mathfrak{g}_1 = \mathrm{Lie}(G_1)$. The differential version of a strict 2-group is known [40] as a strict Lie 2-algebra. These are specified by differential crossed modules, whose definition is given in def. 9.14 on p. 193.

The dg-algebra dual to this 2-algebra is the following:

Example 13.1

Given a differential crossed module $(\mathfrak{g}, \mathfrak{h}, d\alpha, dt)$ (def. 9.14), define a graded vector space

$$V^* = V_1^* \oplus V_2^*$$

by

$$V_1^* \equiv \mathfrak{g}^*$$
$$V_2^* \equiv \mathfrak{h}^*\,,$$

so that

$$\begin{aligned}
\bigwedge^1 V^* &= \mathfrak{g}^* \\
\bigwedge^2 V^* &= \mathfrak{h}^* \oplus \bigwedge^2 \mathfrak{g}^* \\
&\vdots
\end{aligned}$$

Define on $\bigwedge^\bullet V^*$ a differential operator

$$\mathbf{d}^{\mathfrak{g}} \colon \bigwedge^n V^* \to \bigwedge^{(n+1)} V^*$$

by first picking a basis

$$\mathfrak{g}^* = \langle \{\mathbf{a}^a\}_{a=1\ldots \dim(\mathfrak{g})} \rangle \tag{13.1}$$

$$\mathfrak{h}^* = \langle \{\mathbf{b}^A\}_{A=1\ldots \dim(\mathfrak{h})} \rangle \tag{13.2}$$

and then defining the action of $\mathbf{d}^{\mathfrak{g}}$ in that basis as

$$\begin{aligned}
\mathbf{d}^{\mathfrak{g}}\mathbf{a}^a &\equiv -\frac{1}{2} C^a{}_{bc}\mathbf{a}^b\mathbf{a}^c - (dt)^a_A \mathbf{b}^A \\
\mathbf{d}^{\mathfrak{g}}\mathbf{b}^A &\equiv -(d\alpha)^A_{aB}\mathbf{a}^a\mathbf{b}^B \,.
\end{aligned}$$

The various tensor components here are defined in the obvious way as follows:

$$\begin{aligned}
&[t_a, t_b] = C_{ab}{}^c t_c \\
&d\alpha(t_a)(s_B) = (d\alpha)^A_{aB}\, s_A \\
&dt(s_A) = (dt)^a_A t_a \,,
\end{aligned}$$

where $t_a \in \mathfrak{g}$ and $s_A \in \mathfrak{h}$ are elements of the dual basis defined by

$$\begin{aligned}
\mathbf{a}^a(t_b) &= \delta^a_b \\
\mathbf{b}^A(s_B) &= \delta^A_B \,.
\end{aligned}$$

It is straightforward to check that $(\mathbf{d}^{\mathfrak{g}})^2 = 0$ if and only if C, $d\alpha$ and dt define a differential crossed module. In this sense $(\mathbf{d}^{\mathfrak{g}}, \bigwedge^\bullet V^*)$ is the dual incarnation of the differential crossed module $(\mathfrak{g}, \mathfrak{h}, d\alpha, dt)$.

13.3.2 The Source dg-Algebra

The source dg-algebra (representing the path p-groupoid which involves points, paths, surfaces, volumes, etc.) should be defined on a graded vector space V^* such that V_0^* knows about points in U_i, V_1^* about paths in U_i, V_2^* about surfaces, etc.

This motivates the following definition:

Definition 13.10 *For a given patch U_i denote by*

$$\mathfrak{p}_p(U_i) = (\mathbf{d}^{\mathfrak{p}}, \bigwedge W^*)$$

the dg-algebra of level p (def. 13.4) given by

$$W_n^* \equiv \begin{cases} \Gamma(\bigwedge^n T^*U_i) & \textit{for } 0 \leq n \leq p \\ 0 & \textit{otherwise} \end{cases}$$

(where $\Gamma(\bigwedge^n T^*U_i)$ denotes the space of smooth sections of the n-form bundle over U_i) such that for $\omega \in W_n^*$ we have

$$\mathbf{d}^{\mathfrak{p}}\omega \equiv \mathbf{d}\omega \in W_{n+1}^*\,,$$

where $\mathbf{d}$ is the ordinary deRham operator on differential forms over U_i.

Note that by this definition for instance a p-form regarded as an element of W_p^* is *distinguished* from the same p-form regarded as an element of $\bigwedge^p W_1^*$. In particular

$$\mathbf{d}^{\mathfrak{p}} W_p^* = 0$$

since $W_{(p+1)}^* = 0$, while

$$\mathbf{d}^{\mathfrak{p}} \left(\bigwedge^{p} W_1^*\right) \subset \left(\bigwedge^{(p-1)} W_1^*\right) \bigwedge W_2^*$$

need not vanish.

Let us stipulate this family of algebroids $\mathfrak{p}_p$ as the desired source algebroids. This will be justified by the results of the following sections where it is shown that using this family of algebroids the results known from the integral picture are indeed reproduced.

13.3.3 The Connection Morphism

With source and target algebroids in hand, we can now define the morphism between them that shall be addressed as a p-connection, serving as a differential analogue of a holonomy p-functor.

Definition 13.11 *Given an open set $U \subset M$ and a dg-algebra $\mathfrak{g}_p = (\mathbf{d}^{\mathfrak{g}}, \bigwedge^\bullet V^*)$ of level p (def. 13.4) we say that a dg-algebra morphism (def. 13.5)*

$$\mathrm{con}_U \colon \mathfrak{p}_p(U) \to \mathfrak{g}_p$$

is a **local p-connection** *on U if*

- *it is homogenizing (def. 13.7)*
- *it comes from an algebra homomorphism such that*

$$(\mathrm{con}_U)_{\sum_{i=1}^n |v_i|}(v_1 v_2 \cdots v_n) = (\mathrm{con}_U)_{|v_1|}(v_1) \wedge (\mathrm{con}_U)_{|v_2|}(v_2) \wedge \cdots \wedge (\mathrm{con}_U)_{|v_n|}(v_n) \in V^*_{\sum_{i=1}^n |v_i|}\,,$$

where (recall this notation from def. 13.5) $(\mathrm{con}_U)_n$ *is the restriction of* con_U *to* $\bigwedge^n V^*$.

The meaning of the first of these two conditions, and the reason for including it, will become clear when we discuss the *p-curvature* of a p-connection in §13.3.5 (p.348).

The second condition says that for specifying the action of con_U on $\bigwedge^\bullet V^*$ it suffices to know its action on V_0^*, V_1^*, V_2^*, etc.

(So, for instance, the connection morphism con_U of an infinitesimal 1-bundle (to be studied in §13.5 (p.355)) is completely specified by the image $\mathrm{con}_{U_i}(\mathbf{a}^a) = (\mathrm{con}_{U_i})_1(\mathbf{a}^a) = A_i^a \in \Gamma(T^*U_i)$.)

This defines what we want to call a local p-connection, being the differential version of a local holonomy p-functor. In the integral picture we had 1-morphisms between these holonomy p-functors and 2-morphisms between these 1-morphisms, and so on. Similarly, here we have to consider morphisms between local p-connections. Since a p-connection is a chain map, such a morphism is a chain homotopy (def. 13.8). But we are not interested in arbitrary chain homotopoies, but just in those that relate local p-connections that differ by an *infinitesimal* gauge transformation. We will now explain what this is supposed to mean, following and building on the discussion in section 4 of [196].

13.3.4 Infinitesimal n-(gauge)-Transformations

Under infinitesimal transformations between p-connections we want to understand maps between elements of a given fiber of the tangent bundle $T\mathrm{Con}$ of the space Con of all p-connections over some open set U.

Definition 13.12 *If* $\epsilon \mapsto \mathrm{con}_U^{(\epsilon)}$ *is any smooth 1-parameter family of local p-connections, denote by*

$$[\mathrm{con}_U^{(\epsilon)}]$$

the equivalence class of such families under the equivalence relation

$$\mathrm{con}_U^{(\epsilon)} \sim \widetilde{\mathrm{con}}_U^{(\epsilon)} \quad \Leftrightarrow \quad \begin{cases} \mathrm{con}_U^{(0)} = \widetilde{\mathrm{con}}_U^{(0)} \\ \frac{d}{d\epsilon}\mathrm{con}_U^{(\epsilon)}|_{\epsilon=0} = \frac{d}{d\epsilon}\widetilde{\mathrm{con}}_U^{(\epsilon)}|_{\epsilon=0} = \mathrm{con}'_U \end{cases}$$

for some map

$$\mathrm{con}'_U \colon \bigwedge^\bullet V^* \to \bigwedge^\bullet T^*U\,.$$

We write

$$[\mathrm{con}_U^{(\epsilon)}] \equiv [\mathrm{con}_U + \epsilon\,\mathrm{con}'_U]\,.$$

The notion of an infinitesimal (gauge) 1-transformation is now nothing but the concept of a chain homotopy (def. 13.8) applied to con'_U:

Definition 13.13 *An* **infinitesimal (gauge) 1-transformation**

$$\left[\mathrm{con}_U + \epsilon\,\mathrm{con}'_U\right] \xrightarrow{l} \left[\mathrm{con}_U + \epsilon\,\widetilde{\mathrm{con}'_U}\right]$$

between equivalence classes of families of local p-connections (def. 13.12) is a linear map

$$l\colon \bigwedge^{\bullet} V^* \to \bigwedge^{\bullet-1} T^*U$$

such that

$$\widetilde{\mathrm{con}'_U} = \mathrm{con}'_U + [Q, l]\,.$$

So two infinitesimal 1-transformations are **composable** if they correspond to the same con_U and their **composition** is given simply by the sum of their generators:

$$\Big[\mathrm{con}_U + \epsilon\,\mathrm{con}'_U\Big] \xrightarrow{l_1} \Big[\mathrm{con}_U + \epsilon(\mathrm{con}'_U + [Q, l_1])\Big] \xrightarrow{l_2} \Big[\mathrm{con}_U + \epsilon(\mathrm{con}'_U + [Q, l_1 + l_2])\Big]\,.$$

In particular, the infinitesimal **inverse (1-)transformation** to l is its negative $-l$.

With infinitesimal 1-transformations in hand it is easy to recursively define infinitesimal n-transformations as follows:

Definition 13.14 *With respect to an open set $U \subset M$ and a given p-connection* con_U *(def. 13.11) we say that for $n \geq 2$ an* **infinitesimal n-transformation**

$$l_{n-1} \xrightarrow{l_n} \tilde{l}_{n-1}$$

between infinitesimal $(n-1)$-transformations l_{n-1} and $\tilde{l}_{n-1}$ (def. 13.13) is a linear map

$$l_n\colon \bigwedge^{\bullet} V^* \to \bigwedge^{\bullet-n} \Gamma(T^*U)$$

such that

$$\tilde{l}_{n-1} - l_{n-1} = [Q, l_n]\,.$$

So infinitesimal n-transformations of local p-connections are essentially nothing but $(n+1)$-morphisms of dg-algebras (def. 13.9), except that for $n = 1$ we use the infinitesimal notion of morphism as given in def. 13.13.

In the following subsection §13.4 (p.350), nonabelian Deligne hypercohomology will be obtained by assigning such infinitesimal n-morphisms to Čech-n-simplices. In that context it will be very convenient, and is in any case very natural, to call the equivalence class of a p-connection itself a 0-transformation:

Definition 13.15 *An (infinitesimal)* **0-transformation** *is nothing but an element* $[\mathrm{con}_U + \epsilon\,\mathrm{con}'_U]$ *(def. 13.13).*

One easily sees that the maps l_n defining infinitesimal n-transformations cannot be arbitrary linear maps, but have to come from derivations:

13.3.4.1 con-Derivations. Since we required a p-connection con_U to come from an algebra homomorphism (def. 13.11) it is clear that con'_U in def. (13.12) has to be what in def. 6 of [196] is called a con_U*-Leibniz operator of degree 0* and what we here will call a con_U-derivation:

Definition 13.16 *Given any chain map*

$$\Phi\colon \left(\mathbf{d}^B, \bigwedge^{\bullet} B^*\right) \to \left(\mathbf{d}^A, \bigwedge^{\bullet} A^*\right)$$

and a linear map

$$\phi\colon \bigwedge^{\bullet} B^* \to \bigwedge^{\bullet - n} A^*\,,$$

ϕ *is called a* **Φ-derivation of degree** n *if it satisfies the equation*

$$\phi(b_1 b_2) = \phi(b_1)\,\Phi(b_2) + (-1)^{n|b_1|}\Phi(b_1)\,\phi(b_2)\ .$$

In the context of the direct sum complex (def. 13.6), a Φ-derivation is nothing but an ordinary derivation $\bar{\phi}$ composed with $\Phi \simeq \begin{bmatrix} 1 & \Phi \\ 0 & 0 \end{bmatrix}$:

$$\phi = \bar{\phi} \circ \Phi\,.$$

Since Φ is Q-closed it follows that

$$[Q, \phi] = 0 \Leftrightarrow [Q, \bar{\phi}] = 0\,.$$

This means that we can work with Φ-derivations essentially as with ordinary derivations.[14]

It follows immediately that

Proposition 13.1 *The generator l_n of an infinitesimal n-transformation (def. 13.14) with respect to an open set U and a p-connection con_U has to be a con_U-derivation for some $i \in I$ (def. 13.16) of degree n.*

It follows that the l_n are completely specified by defining their action on V^*. We will see examples for this worked out in §13.5 (p.355) and §13.6 (p.359).

Before discussing nonabelian Deligne hypercohomology, at last, we should finish the discussion of local p-connections by mentioning their associated n-curvature:

[14] This relies crucially on the property of Φ to be a chain map, which is the "on-shell" condition discussed in detail in section 4 of [196].

13.3.5 n-Curvature

It turns out that the **curvature** of a p-connection con_U is a measure for its failure to constitute a certain chain map [196, 199]:

We have defined a local p-connection con_U to be a morphism of dg-algebras

$$\mathfrak{p}_p(U) \stackrel{\mathrm{con}_U}{\longrightarrow} \mathfrak{g}_p$$

with source the p-algebroid given by $\mathfrak{p}_p(U)$ and target the dg-algebra $(\mathbf{d}^{\mathfrak{g}}, \bigwedge^\bullet V^*)$. Every such morphism evidently extends to a morphism

$$\mathfrak{dr}(U) \stackrel{\widehat{\mathrm{con}_U}}{\longrightarrow} \mathfrak{g}_p\,,$$

where

$$\mathfrak{dr}(U) \equiv \left(\mathbf{d},\ \bigwedge^{\bullet} \Gamma(T^*U)\right)$$

is simply the de Rham complex on U.

Definition 13.17 *The n-**curvature** $F_U^{(n)}$ of the local p-connection con_U is an $(n{+}1)$-form on U taking values in the vector space V_n defined to be the Q-closure of $\widehat{\mathrm{con}_U}$ at degree n:*

$$F_U^{(n)} \equiv [Q, \widehat{\mathrm{con}_U}]|_{V_n^*} \in \Gamma\left(\bigwedge^{n+1} T^*U,\ V_n\right)\,.$$

Here we used the fact that $[Q, \widehat{\mathrm{con}_U}]$ restricts on V_n^* to a map

$$V_n^* \to \Gamma\left(\bigwedge^{n+1} T^*U\right)\,,$$

so that we can regard $F_U^{(n)}$ as an $(n+1)$-form taking values in the dual V_n of V_n^*.

So for instance if $\{\mathbf{a}^a\}$ is a basis for V_1^* and $\{t_a\}$ is the dual basis, then

$$F_U^{(1)} = \sum_a t_a\,[Q, \widehat{\mathrm{con}_U}](\mathbf{a}^a)\,.$$

Examples will be discussed in more detail in §13.5.1 (p.355) and §13.6.1 (p.359).

The way we have defined con_U it follows that all these curvatures, except that for $n = p$, have to *vanish*:

$$[Q, \mathrm{con}_U] = 0 \quad \Leftrightarrow \quad F_U^{(n)} = 0\,, \quad \text{for } 0 \leq n < p.$$

Had we used $\mathfrak{dr}(U)$ instead of $\mathfrak{p}_p(U)$ as the source p-algebroid and not required con_U to be homogenizing, then also $F_i^{(n=p)}$ would have had to vanish. Recall that the definition of $\mathfrak{p}_p(U)$ was motivated by the observaiton that we needed a dg-algebra of level p (def. 13.4) as a differential analogue of the source p-path p-groupoid.

A physical motivation for why all curvatures except that at top level should vanish can be found in [37]. In §13.6.1 (p.359) we will see that this condition reproduces the condition

found in [92, 31] of the vanishing of the "fake curvature" (*cf.* (11.16), p. 260). In the integral picture this ensures the functoriality of hol_U, while in the differential picture it ensures that con_U is a chain map.

The definition of curvature in terms of Q-closure has two very convenient cosnequences for the formalism:

First, since gauge transformations act on con_i as additive shifts by Q-exact terms, it is immediate that

Proposition 13.2 *The p-curvature $F^{(p)}$ is a globally defined $(p+1)$-form, i.e. there is a global*

$$F^{(p)} \in \Omega^{p+1}(M, V_p)$$

such that every $F_U^{(p)}$ is the restriction of this form to U:

$$F_U^{(p)} = (F^{(p)})|_U \,.$$

Second, to every p-curvature we immediately get a p-Bianchi-identity [199]:

Definition 13.18 *The identity*

$$[Q, F^{(p)}] = 0$$

is called the p-**Bianchi identity**.

13.4 Generalized Deligne Hypercohomology

Motivated by the integral picture of a p-bundle with p-connection (§12 (p.285)) and following the considerations at the beginning of this section, we are interested in associating infinitesimal n-transformations with respect to a p-connection con_U (def. 13.14) to every Čech-n-simplex (def. 13.1) such that their source and target $(n-1)$-morphisms are those associated to the source boundary and target boundary (def. 13.2) of the n-simplex, respectively.

For instance, if $n = 2$ we want to construct diagrams like this:

$$\begin{tikzcd}
 & (l_j)_0 \arrow[dr, "(l_{jk})_1"] & \\
(l_i)_0 \arrow[ur, "(l_{ij})_1"] \arrow[rr, "(l_{ik})_1"'] & \Uparrow (l_{ijk})_2 & (l_k)_0
\end{tikzcd}$$

Here, by def. 13.15, l_0 are equivalence classes of local p-connections. The l_1 are infinitesimal 1-transformations (def. 13.13) between them and the

$$(l_{ijk})_2 \colon (l_{ik})_1 \to (l_{ij})_1 \circ (l_{jk})_1$$

is an infinitesimal 2-transformation (def. 13.14) between these 1-transformations.

This procedure of associating n-transformations to n-simplices is easily formalized and directly leads to the nonabelian Deligne coboundary operator:

13.4.1 The Double Complex of Sheaves of Infinitesimal n-Transformations

Fix a $p \in \mathbb{N}$ and a p-connection (def. 13.11)

$$\mathrm{con} \equiv \mathrm{con}_M \colon \mathfrak{p}_p(M) \to \mathfrak{g}_p$$

on all of M. For any open subset $U \subset M$ write $\mathrm{con}|_U$ for the obvious restriction of con to a p-connection $\mathrm{con}|_U \colon \mathfrak{p}_p(U) \to \mathfrak{g}_p$. These con_U fix fibers in the tangent bundles (def. 13.12) of the spaces of of all p-connections with respect to the subsets $U \subset M$.

For each element $U_i \subset \mathcal{U}$ of a good covering $\mathcal{U}$ let

$$L^n(U_i)$$

be the abelian group (under ordinary addition) of infinitesimal n-transformations (def. 13.14) with respect to $\mathrm{con}|_{U_i}$. So, by prop. (13.1), each $L^n(U_i)$ is a vector space of linear maps

$$l_n \colon \bigwedge^{\bullet} V^* \to \bigwedge^{\bullet - n} \Gamma(T^* U_i)$$

that are $\mathrm{con}|_{U_i}$-derivations (def. 13.16) of degree n.

We then have sheaves $\underline{\mathrm{L}}^n_{\mathrm{con}}$ of infinitesimal n-transformations, defined by

$$\underline{\mathrm{L}}^n_{\mathrm{con}}(U_i) \equiv L^n(U_i) \ .$$

With the operator $[Q,\cdot]$ (def. 13.6) these form a bounded complex of sheaves

$$\underline{\mathrm{L}}^\bullet_{\mathrm{con}} \equiv 0 \to \underline{\mathrm{L}}^p_{\mathrm{con}} \xrightarrow{Q} \underline{\mathrm{L}}^{p-1}_{\mathrm{con}} \xrightarrow{Q} \cdots \xrightarrow{Q} \underline{\mathrm{L}}^0_{\mathrm{con}} \to 0\,,$$

where we decree L^n to have degree $-n$.

Recall the following standard definition of the Čech cochain complex with respect to a given sheaf (*cf.* for instance section 1.3 of [178]):

Definition 13.19 *The* **Čech cochain complex of the sheaf** $\underline{L}^n_{\mathrm{con}}$ *is the complex*

$$0 \to C^0(\mathcal{U}, \underline{L}^n_{\mathrm{con}}) \xrightarrow{\delta} C^1(\mathcal{U}, \underline{L}^n_{\mathrm{con}}) \xrightarrow{\delta} \cdots \xrightarrow{\delta} C^p(\mathcal{U}, \underline{L}^n_{\mathrm{con}}) \to 0$$

with respect to $\mathcal{U}$*, defined as follows:*

The sets here are the cartesian products

$$C^m(\mathcal{U}, \underline{L}^n_{\mathrm{con}}) \equiv \prod_{(i_0,\dots,i_m)\in I^{m+1}} \underline{L}^n_{\mathrm{con}}(U_{i_0\cdots i_m}) \ ,$$

so that an element $\omega^{m,n} \in C^m(\mathcal{U}, \underline{L}^n_{\mathrm{con}})$ *is a map*

$$c \mapsto \omega(c) \in L^n(U_c)$$

that assigns an infinitesimal n*-morphism with respect to* $\mathrm{con}|_{U_{i_0\cdots i_m}}$ *to every Čech* m*-simplex* $c = (i_0 \dots i_m)$*.*

The operator δ *is the dual of the Čech-boundary operator (def. 13.1) denoted by the same symbol, composed with the restriction operation in the sheaf* $\underline{L}^n_{\mathrm{con}}$*. So its action on* $\omega \in C^m(\mathcal{U}, \underline{L}^n_{\mathrm{con}})$ *is*

$$(\delta\omega)(c) \equiv \omega(\delta c)\,|_{U_c}$$

for $c \in C_{m+1}(\mathcal{U})$ *and Čech-*$(m+1)$*-simplex.*

Hence we get a Čech cochain complex assciated to $\underline{\mathrm{L}}^n_{\mathrm{con}}$ for every n. Since the $\underline{\mathrm{L}}^n_{\mathrm{con}}$ form a complex themselves with respect to Q, the result is a double complex. In the standard way (*cf.* [178], p. 28) we hence arrive at the corresponding Čech hypercohomology:

Definition 13.20 *The* r*-th* **generalized (nonabelian) Deligne cohomology** *group with respect to* $\mathcal{U}$ *and* con *is the Čech hypercohomology*

$$\check{H}^r(\mathcal{U}, \underline{L}^\bullet_{\mathrm{con}}) \ ,$$

i.e. the total cohomology of the double complex $C^\bullet(\mathcal{U}, \underline{L}^\bullet_{\mathrm{con}})$

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow Q & & \uparrow Q & & \\
\cdots & \xrightarrow{\delta} & C^m(\mathcal{U}, \underline{L}^{n-1}_{\mathrm{con}}) & \xrightarrow{\delta} & C^{m+1}(\mathcal{U}, \underline{L}^{n-1}_{\mathrm{con}}) & \xrightarrow{\delta} & \cdots \\
 & & \uparrow Q & & \uparrow Q & & \\
\cdots & \xrightarrow{\delta} & C^m(\mathcal{U}, \underline{L}^{n}_{\mathrm{con}}) & \xrightarrow{\delta} & C^{m+1}(\mathcal{U}, \underline{L}^{n}_{\mathrm{con}}) & \xrightarrow{\delta} & \cdots \\
 & & \uparrow Q & & \uparrow Q & & \\
 & & \vdots & & \vdots & &
\end{array}$$

with respect to the **generalized (nonabelian) Deligne coboundary operator**

$$D \equiv \delta + (-1)^m Q\,,$$

which is of **total degee** *1 with respect to the* **total grading**

$$|C^m(\mathcal{U}, \underline{L}^n_{\mathrm{con}})\,| = m - n\,.$$

(Recall that the degree of L^n was defined to be $-n$.)

As usual (e.g. [178], def. 1.3.9) this notion of generalized Deligne cohomology can be made independent of the choice of cover by taking the direct limit over the set of coverings, $\mathcal{U}$, which is ordered under the refinement relation:

$$\check{H}^r(M, \underline{\mathrm{L}}^\bullet_{\mathrm{con}}) \equiv \lim_{\longrightarrow \mathcal{U}} \check{H}^r(\mathcal{U}, \underline{\mathrm{L}}^\bullet_{\mathrm{con}})\ .$$

This is the central definition to be presented here.

Before doing anything with it, let us reassure ourselves that this does incorporate ordinary Deligne cohomology as a special case. (Ordinary Deligne cohomology is introduced in chapter I of [178]. Helpful discussions can be found in section 5.2 of [177] as well as in section 2.2 of [47].)

13.4.1.1 Ordinary Deligne Cohomology. The case that the structure p-algebra is strict and abelian corresponds to

$$\mathbf{d}^{\mathfrak{g}} = 0$$

and

$$V^* = V^*_{n=p}$$

being 1-dimensional.

In this case the generalized Deligne cohomology (def. 13.20) reduces to ordinary Deligne cohomology as follows:

Denote by $\underline{\Omega}^n_M$ the sheaf of n-forms on M.

Since an element in $L^n_{\mathrm{con}}(U)$ maps

$$V^*_{n=p} \to \bigwedge^{p-n} \Gamma(T^*U)$$

we can identify it with a $(p-n)$-form on U and hence we have a surjection

$$\underline{\mathbf{L}}^n_{\mathrm{con}} \xrightarrow{s} \underline{\Omega}^{p-n}_M .$$

Noting that under this map we have

$$s([Q, l_n]) = \mathbf{d}\, s(l_n)$$

it follows that the generalized Deligne double complex reduces to the double complex $C^\bullet(\mathcal{U}, \underline{\Omega}^\bullet_M)$

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow \mathbf{d} & & \uparrow \mathbf{d} & & \\
\cdots & \xrightarrow{\delta} & C^m(\mathcal{U}, \underline{\Omega}^{n+1}_M) & \xrightarrow{\delta} & C^{m+1}(\mathcal{U}, \underline{\Omega}^{n+1}_M) & \xrightarrow{\delta} & \cdots \\
 & & \uparrow \mathbf{d} & & \uparrow \mathbf{d} & & \\
\cdots & \xrightarrow{\delta} & C^m(\mathcal{U}, \underline{\Omega}^{n}_M) & \xrightarrow{\delta} & C^{m+1}(\mathcal{U}, \underline{\Omega}^{n}_M) & \xrightarrow{\delta} & \cdots \\
 & & \uparrow \mathbf{d} & & \uparrow \mathbf{d} & & \\
 & & \vdots & & \vdots & &
\end{array}$$

with respect to the **ordinary Deligne coboundary operator**

$$D \equiv \delta + (-1)^m \mathbf{d} .$$

The hypercohomology of this Čech-de Rham complex is precisely the ordinary Deligne cohomology.

13.4.1.2 Interpretation of Generalized Deligne Hypercohomology. In order to understand how the action of D is related to the task of associating n-transformations to Čech-n-simplices, consider the D-coboundary condition at degree 0, i.e. consider the condition for an element

$$\omega \in C^0(\mathcal{U}, \underline{\mathbf{L}}^\bullet_{\mathrm{con}})$$

to be D-closed:

$$\begin{aligned}
& D\omega(c) = 0 \\
\Leftrightarrow\ & \omega(\delta c) = (-1)^{n+1}[Q, \omega(c)]
\end{aligned}$$

for $c \in C_n(\mathcal{U})$ any Čech-n-simplex and the restriction to U_c on the left being implicit.

Following def. 13.2 we can split the boundary δc of the simplex c into its source part $s(c)$ and its target part $t(c)$ as

$$\delta c = s(c) - t(c) .$$

This allows to equivalently rewrite the above condition as

$$\omega(t(c)) - \omega(s(c)) = (-1)^n [Q, \omega(c)] \,.$$

Comparison of this equation with the definition of an infinitesimal n-transformation of p-connections in def. 13.14 shows that this says nothing but that the n-transformation $\omega(c)$ interpolates between the two $(n-1)$-transformations $\omega(s(c))$ and $\omega(t(c))$:

$$\omega(s(c)) \xrightarrow{\pm\omega(c)} \omega(t(c)) \ .$$

This is precisely what we are interested in.

Therefore it is to be expected that the differential version of the holonomy p-functors $\{\mathrm{hol}_i | i \in I\}$, together with the n-morphisms relating them on multiple overlaps in the integral picture, is given by an element $\omega \in C^0(\mathcal{U}, \underline{\mathrm{L}}^\bullet_{\mathrm{con}})$ that is D-closed.

Different choices of trivializations of the p-bundle correspond to gauge transformations of this data. Hence consider any $\lambda \in C^{-1}(\mathcal{U}, \underline{\mathrm{L}}^\bullet_{\mathrm{con}})$ to be be *any* labeling of Čech-n-simplices by infinitesimal $n+1$-transformations. Since $\omega + D\lambda \in C^0(\mathcal{U}, \underline{\mathrm{L}}^\bullet_{\mathrm{con}})$ is D-closed if ω is, it should be true that the shift

$$\omega \to \omega + D\lambda$$

is the infinitesimal version of a gauge transformation of the holonomy p-functor, i.e. of a natural transformation.

13.4.2 Infinitesimal p-Bundles with p-Connection

All this finally motivates the following two definitions:

Definition 13.21 *An* **infinitesimal p-bundle with p-connection** *on the base manifold M relative to a given global p-connection* $\mathrm{con}\colon \mathfrak{p}_p(M) \to \mathfrak{g}_p$ *is an element of the generalized Deligne cohomology group (def. 13.20)* $\check{H}^0(\mathcal{U}, \underline{L}^\bullet_{\mathrm{con}})$.

Definition 13.22 *A* **local trivialization of an infinitesimal p-bundle with p-connection** *on the base manifold M relative to a given global p-connection* $\mathrm{con}\colon \mathfrak{p}_p(M) \to \mathfrak{g}_p$ *is a representative* $\omega \in C^0(\mathcal{U}, \underline{L}^\bullet_{\mathrm{con}})$ *of an element of the generalized Deligne cohomology group (def. 13.20)* $\check{H}^0(\mathcal{U}, \underline{L}^\bullet_{\mathrm{con}})$.

In the next section it is shown (for $p = 1, 2$) that, indeed, the infinitesimal version of the cocycle relations of a strict principal p-bundle with p-connection (or equivalently of a $(p-1)$-gerbe with connection and curving) is encoded in the condition

$$D\omega = 0 \,, \qquad \omega \in C^0(\mathcal{U}, \underline{\mathrm{L}}^\bullet_{\mathrm{con}}) \ ,$$

and that the infinitesimal version of the effect of gauge transformations (natural transformations of the global holonomy p-functor) is encoded in shifts of the form

$$\omega \to \omega + D\lambda \,, \qquad \lambda \in C^{-1}(\mathcal{U}, \underline{\mathrm{L}}^\bullet_{\mathrm{con}}) \ .$$

13.5 Infinitesimal 1-Bundles with 1-Connection

First we check how ordinary principal bundles with connection look like in the language of dg-algebra morphisms and nonabelian Deligne cohomology. In the next subsection this is then generalized to 2-bundles/gerbes.

13.5.1 The local 1-Connection Morphism

Let the target dg-algebra be the dual of an ordinary Lie (1-)algebra. This is obtained by considering example 13.1 (p. 342) and setting $\mathfrak{h} = 0$.

Define on each $U_i\,, i \in I$ a local 1-connection (def. 13.11) $\mathrm{con}_i \colon \mathfrak{p}_1(U_i) \to \mathfrak{g}_1$ by:

$$\begin{array}{rccc} (\mathrm{con}_i)_1 \;:\; & \bigwedge^1 V^* & \to & \Gamma(T^*U_i) \\ & \mathbf{a}^a & \mapsto & A_i^a \end{array}$$

Recall that we required the connection to be a homogenizing (def. 13.7) morphism of dg-algebras and that $V_n^* = 0$ for $n > 1$ in the algebroid $\mathfrak{p}_1(U_i)$ (def. 13.10). This means that con_i acts trivially on $\bigwedge^n V^*$ for $n \neq 1$.

This map is a chain map (def. 13.5) only if $[Q, \mathrm{con}_i] = 0$ (def. 13.6), which, in the present case, is trivially fulfilled:

$$[Q, \mathrm{con}_i](\mathbf{a}^a) \in V_2^* = 0\,,$$

by the nature of $\mathfrak{p}_1(U_i)$. Recall the discussion in §13.3.5 (p.348) for the relevance of this simple fact. According to the discussion there, the 1-curvature of con_i is

$$\begin{aligned} F_i^{(1)} &= [Q, \widehat{con}_i]|_{V_1^*} \\ &= F_{A_i} \\ &= \mathbf{d}A_i + \frac{1}{2}[A_i, A_i]\,, \end{aligned}$$

as expected.

But according to §13.3.4 (p.345) we are to think of con_i only "infinitesimally" in order to construct an infinitesimal 1-bundle with 1-connection from it. This means that we are to fix a 1-connection

$$\mathrm{con} \equiv \mathrm{con}_M \colon \mathfrak{p}_1(M) \to \mathfrak{g}_1$$

defined on all of M and work only in the fiber of the tangent bundle of the space of all 1-connections over con. So replace con_i by

$$[\mathrm{con} + \epsilon\,\mathrm{con}_i']$$

with con_i' a $\mathrm{con}|_{U_i}$-derivation of degree 0 (def. 13.16).

Writing

$$(\mathrm{con} + \epsilon \mathrm{con}_i')(\mathbf{a}^a) = A_i^a = A^a + \epsilon A_i'^a$$

with $A^a \in \Gamma(T^*M)$ and $A_i'^a \in \Gamma(T^*U_i)$ we get the curvature

$$F_i^{(1)} = \mathbf{d}A + \frac{1}{2}[A, A] + \mathbf{d}A_i' + \epsilon[A_i', A']\,.$$

13.5.2 Infinitesimal (Gauge) 1-Transformations

Now consider an infinitesimal 1-transformation (def. 13.13)

$$\left[\mathrm{con} + \epsilon\,\mathrm{con}'_i\right] \xrightarrow{l} \left[\mathrm{con} + \epsilon\left(\mathrm{con}'_i + [Q, l]\right)\right].$$

According to proposition 13.1 (p. 347) its generator

$$l\colon \bigwedge^{\bullet} V^* \to \Gamma\left(\bigwedge^{\bullet-1} T^*U_i\right)$$

has to be a $\mathrm{con}|_{U_i}$-derivation of degree one and is hence completely defined by setting

$$l(\mathbf{a}^a) \equiv -(\ln h)^a \in C^\infty(U_i)\ .$$

Its Q-closure is therefore

$$\begin{aligned}
[Q, l](\mathbf{a}^a)\,, &= \mathbf{d}l(\mathbf{a}^a) + l(\mathbf{d}^{\mathfrak{g}}\mathbf{a}^a) \\
&= -\mathbf{d}(\ln h)^a + l\left(-\frac{1}{2}C^a{}_{bc}\mathbf{a}^b\mathbf{a}^c\right) \\
&= -\mathbf{d}(\ln h)^a - [A, \ln(h)]^a\ .
\end{aligned}$$

This is the infinitesimal version of the usual gauge transformation

$$\tilde{A}_i = hA_ih^{-1} + h\mathbf{d}h^{-1}\,.$$

Note how the operator Q takes care that the nonabelian term $[A, \ln(h)]$ appears.

13.5.3 Cocycle Relations

On double intersections, U_{ij}, let $\mathrm{con}'_i|_{U_{ij}}$ and $\mathrm{con}'_j|_{U_{ij}}$ be related by infinitesimal 1-transformations (def. 13.13)

$$[\mathrm{con} + \epsilon\,\mathrm{con}'_i]|_{U_{ij}} \xrightarrow{\mathfrak{g}_{ij}} [\mathrm{con} + \epsilon\,\mathrm{con}'_j]|_{U_{ij}}\ ,$$

i.e.

$$\mathrm{con}'_j|_{U_{ij}} = \mathrm{con}'_i|_{U_{ij}} + [Q, \mathfrak{g}_{ij}]\,,$$

given by

$$\mathfrak{g}_{ij}(\mathbf{a}^a) = -(\ln g_{ij})^a \in C^\infty(U_{ij})\ .$$

Require that on triple overlaps U_{ijk} the diagram

$$\begin{tikzcd}
 & {[\mathrm{con}_j]} \arrow[dr, "\mathfrak{g}_{jk}"] & \\
{[\mathrm{con}_i]} \arrow[ur, "\mathfrak{g}_{ij}"] \arrow[rr, "\mathfrak{g}_{ik}"'] & & {[\mathrm{con}_k]}
\end{tikzcd}$$

commutes. This says that

$$\mathfrak{g}_{ij} + \mathfrak{g}_{jk} = \mathfrak{g}_{ik} .$$

In components this means that

$$\ln(g_{ij}) + \ln(g_{ij}) = \ln(g_{ik}) ,$$

which is the infinitesimal version of the familiar cocycle condition

$$g_{ij} g_{jk} = g_{ik} .$$

13.5.4 Hypercohomology Description

We now rederive the above considerations using nonabelian Deligne hypercohomology as defined in §13.4 (p.350):

The nonabelian 1-bundle is described by the element $\omega \in \check{H}^0(\mathcal{U}, \underline{\mathrm{L}}^\bullet_{\mathrm{con}})$ given by

$$\begin{aligned} \omega(i) &= \mathrm{con} + \mathrm{con}'_i \\ \omega(ij) &= \mathfrak{g}_{ij} . \end{aligned}$$

The condition that this be D-closed yields

- the condition that the connection con_i be a chain map:

$$\begin{aligned} 0 &= (D\omega)(i) \\ &= \underbrace{(\delta\omega)(i)}_{=0} + (Q\omega)(i) \\ &= [Q, \omega(i)] \\ &= [Q, \mathrm{con} + \epsilon\, \mathrm{con}'_i] \end{aligned}$$

- the condition that $\mathfrak{g}_{ij}$ is the 1-transformation relating con_i with con_j:

$$\begin{aligned} 0 &= (D\omega)(ij) \\ &= (\delta\omega)(ij) + (Q\omega)(ij) \\ &= \omega(j) - \omega(i) - [Q, \omega(ij)] \\ &= \mathrm{con}'_j - \mathrm{con}'_i - [Q, \mathfrak{g}_{ij}] \end{aligned}$$

  (Note again that this is formally precisely as in the abelian case, but that Q correctly incorporates the nonabelian aspects, as discussed in §13.5.2 (p.356)).

- the cocycle condition for $\mathfrak{g}_{ij}$:

$$\begin{aligned} 0 &= (D\omega)(ijk) \\ &= (\delta\omega)(ijk) + (Q\omega)(ijk) \\ &= -\omega(jk) + \omega(ik) - \omega(ij) + [Q, \omega(ijk)] \\ &= \mathfrak{g}_{ik} - \mathfrak{g}_{ij} - \mathfrak{g}_{jk} - 0 . \end{aligned}$$

Now let $\lambda \in \Omega_{-1}$ and consider the Deligne gauge transformation

$$\omega \to \omega + D\lambda$$

where

$$\lambda(i) = \left(\mathbf{a}^a \mapsto -(\ln h_i)^a\right) .$$

We have

$$\begin{aligned} D\lambda(i) &= [Q, \lambda(i)] \\ D\lambda(ij) &= \lambda(i) - \lambda(j) - 0 \end{aligned}$$

and hence

$$\begin{aligned} \mathrm{con}'_i &\to \mathrm{con}'_i + [Q, \lambda(i)] \\ \mathfrak{g}_{ij} &\to \mathfrak{g}_{ij} + \lambda(i) - \lambda(j) \end{aligned}$$

which implies

$$\begin{aligned} A_i &\to A_i - \mathbf{d}_{A_i} \ln h_i - [A_i, \ln h_i] \\ \ln g_{ij} &\to \ln g_{ij} + \ln h_i - \ln h_j , \end{aligned}$$

corresponding to the finite gauge transformations

$$\begin{aligned} A_i &\to h_i A_i h_i^{-1} + h_i \mathbf{d} h_i^{-1} \\ g_{ij} &\to h_i g_{ij} h_j^{-1} . \end{aligned}$$

This shows that nonabelian Deligne hypercohomology completely captures the infinitesimal version of the cocycle conditions of locally trivialized nonabelian principal (1-)bundles with (1-)connections, including the rules for gauge transformations relating different local trivializations.

13.6 Strict Infinitesimal 2-Bundles with 2-Connection

With just a little more work the above discussions of infinitesimal 1-bundles generalizes to that of 2-bundles. The reader should compare the infinitesimal cocycle relations which we obtain with their finite version listed in Prop. 11.2 (p. 260).

13.6.1 The local 2-Connection Morphism

Let the target dg-algebra $\mathfrak{g}_2$ be the dual of a strict Lie 2-algebra as in example 13.1. Define on each U_i a local 2-connection $\mathrm{con}_i \colon \mathfrak{p}_2(U_i) \to \mathfrak{g}_2$ by the following maps:

$$\begin{aligned} \mathrm{con}_i(\mathbf{a}^a) &= A_i^a \in \Gamma(T^*U_i) \\ \mathrm{con}_i\big(\mathbf{b}^A\big) &= B_i^A \in \Gamma\left(\bigwedge^2 T^*U_i\right) . \end{aligned}$$

This defines a chain map only if $[Q, \mathrm{con}_i] = 0$, which is the case if

$$\begin{aligned} 0 &= \mathbf{d}\mathrm{con}(\mathbf{a}^a) - \mathrm{con}(\mathbf{d}^{\mathfrak{g}}\mathbf{a}^a) \\ &= \mathbf{d}A_i^a + \frac{1}{2}C^a{}_{bc}A_i^b \wedge A_i^c + dt^a_A B_i^A \\ &= (F_{A_i} + dt(B_i))^a . \end{aligned}$$

This is the vanishing of the 1-curvature (def. 13.17) of the infinitesimal 2-bundle (*cf.* §13.3.5 (p.348)). The expression

$$F_i^{(1)} = F_{A_i} + dt(B_i)$$

has been called the *fake curvature* in [49]. It has been found in [92, 31] that the condition $F_i^{(1)} = 0$ is a necessary requirement for hol_i to be a 2-functor. Here it is, analogously, a necessary condition for con_i to be a morphism of dg-algebras.

Note that there is no condition coming from

$$0 = [Q, \mathrm{con}_i]\big(\beta^A\big) \in W_3^* ,$$

since W_3^* of $\mathfrak{p}_2(U_i)$ is trivial, by def. 13.10. The 2-curvature is

$$\begin{aligned} F_i^{(2)} &= [Q, \widehat{\mathrm{con}_i}]|_{V_2^*} \\ &= \mathbf{d}_{A_i} B_i \\ &= \mathbf{d}B_i + d\alpha(A_i)(B_i) , \end{aligned}$$

which is in general non-vanishing.

As done for 1-bundles in the previous section, we should really consider vectors in the tangent bundle to all 2-connections. Fix a 2-connection

$$\mathrm{con} \equiv \mathrm{con}_m \colon \mathfrak{p}_2(M) \to \mathfrak{g}_2$$

and replace the above con_i by

$$[\mathrm{con} + \epsilon\, \mathrm{con}_i']$$

with con_i' a $\mathrm{con}|_{U_i}$-derivation of degree 0 (def. 13.16). We write

$$\begin{aligned} (\mathrm{con} + \epsilon\, \mathrm{con}_i')(\mathbf{a}^a) &= A^a|_{U_i} + \epsilon\, A_i'^a \\ (\mathrm{con} + \epsilon\, \mathrm{con}_i')\big(\mathbf{b}^A\big) &= B^A|_{U_i} + \epsilon\, B_i'^A . \end{aligned}$$

13.6.2 Infinitesimal (Gauge) 1-Transformations

An infinitesimal 1-gauge transformation (def. 13.13)

$$\left[\mathrm{con} + \epsilon\,\mathrm{con}'_i\right] \xrightarrow{\ l\ } \left[\mathrm{con} + \epsilon\,\mathrm{con}'_i + \epsilon[Q,l]\right]$$

is specified by

$$\begin{aligned} l(\mathbf{a}^a) &= -(\ln h)^a \in C^\infty(M) \\ l\big(\mathbf{b}^A\big) &= a^A \in \Gamma(T^*M)\ . \end{aligned}$$

The Q-closure of this map is given by

$$\begin{aligned} [Q,l](\mathbf{a}^a) &= \mathbf{d}l(\mathbf{a}^a) + l(\mathbf{d}^{\mathfrak{g}}\mathbf{a}^a) \\ &= -\mathbf{d}(\ln h)^a + l\bigg(-\frac{1}{2}C^a{}_{bc}\mathbf{a}^b\mathbf{a}^c - dt^a_A\mathbf{b}^A\bigg) \\ &= -\mathbf{d}(\ln h)^a - [A,(\ln h)]^a - dt(a)^a \end{aligned}$$

and

$$\begin{aligned} [Q,l]\big(\mathbf{b}^A\big) &= \mathbf{d}l\big(\mathbf{b}^A\big) + l\big(\mathbf{d}^{\mathfrak{g}}\mathbf{b}^A\big) \\ &= \mathbf{d}a^A + l\big(-d\alpha^A_{aB}\mathbf{a}^a\mathbf{b}^B\big) \\ &= d\alpha(\ln h)(B)^A + \mathbf{d}a^A + d\alpha(A)(a)^A\ . \end{aligned}$$

This is the infinitesimal version of

$$\begin{aligned} A &\to hAh^{-1} + h\mathbf{d}h^{-1} - dt(a) \\ B &\to \alpha(h)(B) + \mathbf{d}_A a + a \wedge a\,. \end{aligned}$$

13.6.3 Cocycle Relations

On each U_{ij} let there be an infinitesimal 1-transformation

$$\left[\mathrm{con} + \epsilon\,\mathrm{con}'_i\right] \xrightarrow{\ \mathfrak{g}_{ij}\ } \left[\mathrm{con} + \epsilon\,\mathrm{con}'_j\right]$$

given by

$$\begin{aligned} \mathfrak{g}_{ij}(\mathbf{a}^a) &= -\ln(g_{ij})^a \\ \mathfrak{g}_{ij}\big(\mathbf{b}^A\big) &= a^A_{ij}\,. \end{aligned}$$

The differential version of the diagram on p. 259, which expresses how the 2-holonomy is gauge transformed when going from U_i to U_j to U_k, is the diagram

$$\begin{tikzcd} & {[\mathrm{con}_j]} \arrow[dr, "\mathfrak{g}_{jk}"] & \\ {[\mathrm{con}_i]} \arrow[ur, "\mathfrak{g}_{ij}"] \arrow[rr, "\mathfrak{g}_{ik}"'] \arrow[r, phantom, shift left=4ex, "\Uparrow\,\mathfrak{f}_{ijk}"] & & {[\mathrm{con}_k]} \end{tikzcd}$$

This says that there is an infinitesimal 2-transformation (def. 13.14) given by

$$\mathfrak{f}_{ijk} \colon \bigwedge^{\bullet} V^* \to \bigwedge^{\bullet - 2} T^* U_{ijk}$$

$$\mathfrak{f}_{ijk}\big(\mathbf{b}^A\big) \; = -(\ln f_{ijk})^A \in C^\infty(U_{ijk})$$

going between the 1-gauge transformations $\mathfrak{g}_{ij} \circ \mathfrak{g}_{jk}$ and $\mathfrak{g}_{ik}$. This means that the equation

$$\mathfrak{g}_{ij} + \mathfrak{g}_{jk} = \mathfrak{g}_{ik} + [Q, \mathfrak{f}_{ijk}]$$

holds.

In terms of components this equation implies that

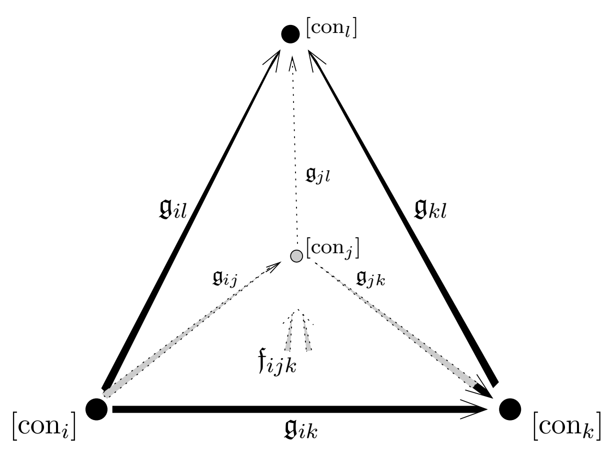
$$\ln g_{ij} + \ln g_{jk} = \ln g_{ik} + dt(\ln f_{ijk})$$

and

$$a_{ij} + a_{jk} = a_{ik} - \mathbf{d}_{A_i} \ln f_{ijk} \, .$$

These are indeed the infinitesimal versions of the known cocycylce conditions for nonabelian 2-bundles/gerbes.

The $\mathfrak{f}_{ijk}$ have to satisfy a law saying that the 3-morphism inside this tetrahedron is trivial:



There are $\mathfrak{f}$s labeling all the four faces of this tetrahedron, but for readability only one of them has been indicated. When the tetrahedron is flattened out we can write

$$
\begin{tikzcd}
{[\mathrm{con}_j]} \arrow[r, "\mathfrak{g}_{jk}"] \arrow[dr, phantom, near start, "\Nwarrow \mathfrak{f}_{ijk}"] & {[\mathrm{con}_k]} \arrow[d, "\mathfrak{g}_{kl}"] \\
{[\mathrm{con}_i]} \arrow[u, "\mathfrak{g}_{ij}"] \arrow[ur, "\mathfrak{g}_{ik}"] \arrow[r, "\mathfrak{g}_{il}"'] & {[\mathrm{con}_l]} \arrow[ul, phantom, near start, "\Uparrow \mathfrak{f}_{ikl}"]
\end{tikzcd}
=
\begin{tikzcd}
{[\mathrm{con}_j]} \arrow[r, "\mathfrak{g}_{jk}"] \arrow[dr, "\mathfrak{g}_{jl}"'] & {[\mathrm{con}_k]} \arrow[d, "\mathfrak{g}_{kl}"] \arrow[dl, phantom, near start, "\Nearrow \mathfrak{f}_{jkl}"] \\
{[\mathrm{con}_i]} \arrow[u, "\mathfrak{g}_{ij}"] \arrow[r, "\mathfrak{g}_{il}"'] \arrow[ur, phantom, near start, "\Uparrow \mathfrak{f}_{ijl}"] & {[\mathrm{con}_l]}
\end{tikzcd}
$$

which implies (for infinitesimal transformations)

$$\mathfrak{f}_{jkl} - \mathfrak{f}_{ikl} + \mathfrak{f}_{ijl} - \mathfrak{f}_{ijk} = 0\,.$$

In components this says that

$$\ln f_{jkl} - \ln f_{ikl} + \ln f_{ijl} - \ln f_{ijk} = 0\,.$$

This is once again the correct infinitesimal version of the respective cocycle condition in a nonabelian 2-bundle/gerbe.

Of course for this last relation all effects of the nonabelian nature of $\mathfrak{g}$ and $\mathfrak{h}$ are of higher order and hence do not appear here. So the above equation is of the same form as in the abelian case.

13.6.4 Hypercohomology Description

We now rederive all the above considerations concerning 2-bundles with 2-connection using nonabelian Deligne hypercohomology (*cf.* §13.4 (p.350)):

The nonabelian 2-bundle is described by the element $\omega \in \check{H}^0(\mathcal{U}, \underline{\mathrm{L}}^\bullet_{\mathrm{con}})$ given by

$$
\begin{aligned}
\omega(i) &= \mathrm{con} + \epsilon\, \mathrm{con}'_i \\
\omega(ij) &= \mathfrak{g}_{ij} \\
\omega(ijk) &= \mathfrak{f}_{ijk}\,.
\end{aligned}
$$

the condition that this be D-closed yields

- the condition that the connection con_i be a chain map:

$$
\begin{aligned}
0 &= (D\omega)(i) \\
&= \underbrace{(\delta\omega)(i)}_{=0} + (Q\omega)(i) \\
&= [Q, \omega(i)] \\
&= [Q, \mathrm{con} + \epsilon\, \mathrm{con}'_i]
\end{aligned}
$$

- the condition that $\mathfrak{g}_{ij}$ is the 1-transformation relating con_i with con_j:

$$\begin{aligned} 0 &= (D\omega)(ij) \\ &= (\delta\omega)(ij) + (Q\omega)(ij) \\ &= \omega(j) - \omega(i) - [Q, \omega(ij)] \\ &= \mathrm{con}'_j - \mathrm{con}'_i - [Q, \mathfrak{g}_{ij}] \end{aligned}$$

  (Note again that this is formally precisely as in the abelian case, but that $[Q, \mathfrak{g}_{ij}]$ correctly incorporates the nonabelian aspects and would also incorporate weak aspects, if included, as discussed in §13.6.2 (p.360)).

- the cocycle condition for $\mathfrak{g}_{ij}$:

$$\begin{aligned} 0 &= (D\omega)(ijk) \\ &= (\delta\omega)(ijk) + (Q\omega)(ijk) \\ &= -\omega(jk) + \omega(ik) - \omega(ij) + [Q, \omega(ijk)] \\ &= \mathfrak{g}_{ik} + [Q, \mathfrak{f}_{ijk}] - \mathfrak{g}_{ij} - \mathfrak{g}_{jk}\,. \end{aligned}$$

- the coherence law for $\mathfrak{f}_{ijk}$:

$$\begin{aligned} 0 &= (D\omega)(ijkl) \\ &= (\delta\omega)(ijkl) - (Q\omega)(ijkl) \\ &= -\omega(jkl) + \omega(ikl) - \omega(ijl) + \omega(ijk) \\ &= -\mathfrak{f}(jkl) + \mathfrak{f}(ikl) - \mathfrak{f}(ijl) + \mathfrak{f}(ijk) \end{aligned}$$

Now let $\lambda \in \Omega_{-1}$ and denote the components of λ as follows:

$$\begin{aligned} \lambda(i) &= \begin{pmatrix} \mathbf{a}^a \mapsto -(\ln h_i)^a \\ \mathbf{b}^A \mapsto \alpha_i^A \end{pmatrix} \\ \lambda(ij) &= \begin{pmatrix} \mathbf{a}^a \mapsto 0 \\ \mathbf{b}^A \mapsto -(\ln p_{ij})^A \end{pmatrix}\,. \end{aligned}$$

Consider the Deligne gauge transformation

$$\omega \to \omega + D\lambda\,.$$

We have

$$\begin{aligned} D\lambda(i) &= [Q, \lambda(i)] \\ D\lambda(ij) &= \lambda(j) - \lambda(i) - [Q, \lambda(ij)] \\ D\lambda(ijk) &= \lambda(ik) - \lambda(ij) - \lambda(jk) + [Q, \lambda(ijk)]\,. \end{aligned}$$

Hence

$$\begin{aligned} \mathrm{con}'_i &\to \mathrm{con}'_i + [Q, \lambda(i)] \\ \mathfrak{g}_{ij} &\to \mathfrak{g}_{ij} + \lambda(i) - \lambda(j) + [Q, \lambda(ij)] \\ \mathfrak{f}_{ijk} &\to \mathfrak{f}_{ijk} + \lambda_{ik} - \lambda_{ij} - \lambda_{jk} + [Q, \lambda_{ijk}]\,, \end{aligned}$$

which in components yields

$$
\begin{aligned}
A_i &\to A_i - \mathbf{d}_{A_i} \ln h_i + dt(\alpha_i) \\
B_i &\to B_i + d\alpha(\ln h_i)(B_i) + \mathbf{d}_{A_i}\alpha_i \\
\ln g_{ij} &\to \ln g_{ij} + \ln h_i - \ln h_j + dt(p_{ij}) \\
a_{ij} &\to a_{ij} + \alpha_i - \alpha_j - \mathbf{d}_A \ln p_{ij} \\
\ln f_{ijk} &\to \ln f_{ijk} + \ln p_{ik} - \ln p_{ij} - \ln p_{jk} \,.
\end{aligned}
$$

These can be checked to be the infinitesimal version of the respective finite gauge transformation in a 2-bundle/gerbe.

In conclusion this demonstrates that nonabelian Deligne cohomology correctly captures the infinitesimal version of the cocycle relations and gauge transformations of a nonabelian gerbe or 2-bundle with 2-connection (*cf.* Prop. 11.2, p. 260). Note however that the condition of vanishing fake curvature (*cf.* p. 359) has not been obtained in the context of nonabelian gerbes. As we have discussed, this condition is related to the existence of a notion of 2-holonomy, which has so far not been discussed for nonabelian gerbes.

13.7 Semistrict Infinitesimal $\mathfrak{g}_k$-2-Bundles with 2-Connection

After having convinced ourselves in §13.5 (p.355) and §13.6 (p.359) that known results about strict 1- and 2-bundles with connection are reproduced by nonabelian Deligne hypercohomology, let us now apply the formalism to something new.

The differential formalism can handle all semistrict target Lie p-algebras (as well as semistrict Lie p-algebroids), no matter if these are integrable to Lie p-groups or not. This allows for instance to consider infinitesimal 2-bundles with target algebra coming from the family $\mathfrak{g}_k$ of non-strict Lie 2-algebras that were introduced in [40]. The 2-algebras $\mathfrak{g}_k$ are non-strict only very slightly and still give rise to very rich structures, as shown in [32].

13.7.1 The local 2-Connection Morphism

Let $\mathfrak{g}$ be any semisimple Lie algebra and let $k \in \mathbb{R}$ be any real number. Then the dg-algebra $(\mathbf{d}^{\mathfrak{g}}, \bigwedge^\bullet V^*)$ dual to the semistrict Lie 2-algebra $\mathfrak{g}_k$ is given by $V^* = V_1^* \oplus V_2^*$ with

$$
\begin{aligned}
V_1^* &= \mathfrak{g}^* = \langle \{\mathbf{a}^a\}_{a=1,\dots\dim(\mathfrak{g})}\rangle \\
V_2^* &= (\mathrm{Lie}(\mathbb{R}))^* = \langle \{\mathcal{B}\}\rangle \,,
\end{aligned}
$$

where we have chosen a basis $\{\mathbf{a}^a\}_a$ for V_1^* and a basis $\{\mathcal{B}\}$ for V_2^*. In this basis the action of $\mathbf{d}^{\mathfrak{g}}$ is given by

$$
\begin{aligned}
\mathbf{d}^{\mathfrak{g}}\mathbf{a}^a &\equiv -\frac{1}{2}C^a{}_{bc}\mathbf{a}^b\mathbf{a}^c \\
\mathbf{d}^{\mathfrak{g}}\mathcal{B} &\equiv -\mu_{abc}\mathbf{a}^a\mathbf{a}^b\mathbf{a}^c \,.
\end{aligned}
$$

As before in example 13.1 the tensor C comes from the structure constants of $\mathfrak{g}$. The tensor μ on the other hand is defined by

$$
\mu_{abc} \equiv k\langle t_a, [t_b, t_c]\rangle \,,
$$

where $\{t_a\}$ is the basis dual to $\{\mathbf{a}^a\}$. When we use the Killing form $\langle\cdot,\cdot\rangle$ to lower indices we can equivalently write

$$
\begin{aligned}
\mathbf{d}^{\mathfrak{g}}\mathbf{a}^a &\equiv -\frac{1}{2}C^a{}_{bc}\mathbf{a}^b\mathbf{a}^c \\
\mathbf{d}^{\mathfrak{g}}\mathcal{B} &\equiv -\frac{k}{2}\,C_{abc}\mathbf{a}^a\mathbf{a}^b\mathbf{a}^c \,.
\end{aligned}
$$

Since $C^a{}_{b[c}C^b{}_{de]} = 0$ (the Jacobi identity in $\mathfrak{g}$) this makes the nilpotency of $\mathbf{d}^{\mathfrak{g}}$ manifest.

We can now closely follow the discussion of strict infinitesimal 2-bundles in §13.6 (p.359).

On U_i the 2-connection is determined by

$$
\begin{aligned}
\mathrm{con}_i(\mathbf{a}^a) &= A_i^a \in \Gamma(T^*U_i) \\
\mathrm{con}_i(\mathcal{B}) &= B_i \in \Gamma\left(\bigwedge^2 T^*U_i\right) \,.
\end{aligned}
$$

The condition for con_i to be Q-closed says that A_i must be *flat*:

$$\begin{aligned} 0 &= \mathbf{d}\mathrm{con}_i(\mathbf{a}^a) - \mathrm{con}_i(\mathbf{d}^{\mathfrak{g}}\mathbf{a}^a) \\ &= \mathbf{d}A_i^a + \frac{1}{2}C^a{}_{bc}A_i^b \wedge A_i^c \\ &= F_{A_i}\,. \end{aligned}$$

This is the vanishing of the fake curvature known from §13.6.1 (p.359), only that here $dt = 0$, so that the fake curvature is the same as the ordinary curvature of A_i.

The 2-curvature of con_i (*cf.* §13.3.5 (p.348)) is

$$\begin{aligned} F^{(2)}|_{U_i} &= [Q, \widehat{\mathrm{con}_i}]|_{V_2^*} \\ &= \mathbf{d}\widehat{\mathrm{con}_i}(\mathcal{B}) - \widehat{\mathrm{con}_i}(\mathbf{d}^{\mathfrak{g}}\mathcal{B}) \\ &= \mathbf{d}B_i + k\,\langle A_i, [A_i, A_i]\rangle\,. \end{aligned}$$

Note that $\langle A_i, [A_i, A_i]\rangle$ is proportional to the Chern-Simons form of the flat connection A_i. Furthermore note that the 2-Bianchi-identity (def. 13.18) is

$$0 = \mathbf{d}F_i^{(2)} \propto \langle \mathbf{d}A_i, [A_i, A_i]\rangle\,,$$

This expression does indeed vanish since $\mathbf{d}A_i = -\frac{1}{2}[A_i, A_i]$ for flat A_i, so that

$$\mathbf{d}F_i^{(2)} \propto \langle [A_i, A_i], [A_i, A_i]\rangle = 0$$

due to the Bianchi identity.[15]

13.7.2 Cocycle and Gauge Transformation Relations

Now let us study the cocycle conditions and gauge transformation laws of infinitesimal $\mathfrak{g}_k$-2-bundles. Recall that, as in the discussion of strict 2-bundles §13.6 (p.359), we have

$$\begin{aligned} A_i &= A|_{U_i} + \epsilon A_i' \\ B_i &= B|_{U_i} + \epsilon B_i' \end{aligned}$$

with A and B globally defined.

With a 1-transformation as in §13.5.2 (p.356)

$$\mathrm{con}_i \xrightarrow{\mathfrak{g}_{ij}} \mathrm{con}_j$$

we have on U_{ij} the relations

$$\begin{aligned} A_i &= A_j - \mathbf{d}\ln g_{ij} - [A_j, (\ln g_{ij})] \\ B_i &= B_j + \mathbf{d}a_{ij} + 3k\,\langle \ln g_{ij}, [A_i, A_i]\rangle\,. \end{aligned}$$

[15] This should be closely related to the fact (indicated at the end of [32]) that 2-bundles with structure algebra the strict 2-algebra $\mathcal{P}_k\mathfrak{g}$ (which is *equivalent* to $\mathfrak{g}_k$) can be obtained from an ordinary G-bundle only if the first Pontryagin class $p_1/2 = \frac{1}{8\pi^2}\langle F_A, F_A\rangle$ vanishes.

(Recall that A is the globally defined 1-form around which the connection of the infinitesimal 2-bundle is the "perturbation", $A_i = A|_{U_i} + \epsilon A'_i$. See §13.3.4 (p.345).)

On triple intersections U_{ijk} a 2-transformation

$$\mathfrak{g}_{ik} \xrightarrow{\mathfrak{f}_{ijk}} \mathfrak{g}_{ij} + \mathfrak{g}_{jk}$$

as in §13.6.3 (p.360) implies the laws

$$\begin{aligned} \ln g_{ij} + \ln g_{jk} - \ln g_{ik} &= 0 \\ a_{ij} + a_{jk} &= a_{ik} - \mathbf{d} \ln f_{ijk} \,. \end{aligned}$$

These last two are *independent* of the deformation parameter k. (Because $\mathfrak{f}_{ijk}$, being a 2-transformation, only acts nontrivially on $\mathcal{B}$, while $\mathbf{d}^{\mathfrak{g}}$ doesn't produce any copies of $\mathcal{B}$.) So the transformation laws for A_i and for $\ln g_{ij}$ are exactly those of an ordinary G-bundle without any twists introduced by k.

Similarly following the previous discussion in §13.6.4 (p.362) one finds that gauge transformations are described by the following equations:

$$\begin{aligned} A_i &\to A_i - \mathbf{d}_{A_i} \ln h_i \\ \ln g_{ij} &\to \ln g_{ij} + \ln h_i - \ln h_j \end{aligned}$$

and

$$\begin{aligned} B_i &\to B_i + \mathbf{d}\alpha_i + 3k \,\langle \ln h_i, [A, A] \rangle \\ a_{ij} &\to a_{ij} + \alpha_i - \alpha_j - \mathbf{d} \ln p_{ij} \\ \ln f_{ijk} &\to \ln f_{ijk} + \ln p_{ik} - \ln p_{ij} - \ln p_{jk} \,. \end{aligned}$$

13.7.3 Generalized Deligne Cohomology Classes

One can now in principle work out the Deligne cohomology classes for $\mathfrak{g}_k$-2-bundles. These are classes of those sets $\{A_i, B_i, \ln g_{ij}, a_{ij} \ln f_{ijk}\}$ that satisfy the above cocycle conditions, divided out by the above gauge transformations.

We will not give a full discussion of this issue here, but make the following comments:

The conditions on $\{A_i, \ln g_{ij}\}$ are those of a principal $\exp(G)$-bundle with a flat connection. Any such bundle has to be trivial and hence it is always possible to go to a gauge (choose a local trivialization) in which $\ln g_{ij} = 0$ identically. In this gauge then the cocycle condition for B_i becomes

$$B_i = B_j + \mathbf{d}a_{ij}$$

as in a strict abelian 2-bundle/gerbe.

Once this gauge has been fixed the gauge parameter $\ln h_i$ is restricted to satisfy $\ln h_i = \ln h_j$ on double overlaps U_{ij}. Hence it must be a globally defined function

$$\ln h \colon M \to \mathfrak{g} \,.$$

This again implies that the gauge transformation law for B_i becomes that of an abelian gerbe up to a global shift:

$$B_i \to B_i + \mathbf{d}\alpha_i + 3k \langle \ln h, [A, A] \rangle$$

The other components, a_{ij} and f_{ijk}, satisfy the unmodified cocycle relations and gauge transformation laws of an abelian gerbe without any appearance of k.

So it seems one should look at classes of pairs

$$(A, \mathcal{G} = (f_{ijk}, a_{ij}, B_i))$$

with $A \in \Omega^1(M, \mathfrak{g})$ a 1-form on $\mathfrak{g}$ satisfying $\mathbf{d}A + \frac{1}{2}[A, A] = 0$ and $\mathcal{G}$ an abelian gerbe on M. It seems like we need to identify two such pairs $(A, \mathcal{G} = (f_{ijk}, a_{ij}, B_i))$ and $(\tilde{A}, \tilde{\mathcal{G}} = (\tilde{f}_{ijk}, \tilde{a}_{ij}, \tilde{B}_i))$ iff $\mathcal{G}$ and $\tilde{\mathcal{G}}$ have representatives such that

$$\begin{aligned} f_{ijk} &= \tilde{f}_{ijk} \\ a_{ij} &= \tilde{a}_{ij} \end{aligned}$$

and

$$\mathbf{d}B + k\langle A, [A, A]\rangle = \mathbf{d}\tilde{B} + k\langle \tilde{A}, [\tilde{A}, \tilde{A}]\rangle \,.$$

While this needs to be better understood, this discussions shows that in principle the "infinitesimal formalism" including nonabelian Deligne hypercohomology allows to address issues that cannot be addressed at all in the "integral formalism".

13.8 Strict Infinitesimal 3-Bundles with 3-Connection

Finally, we can check that the differential formalism correctly reproduces what is known about 3-bundles with 3-connection.

Let the target dg-algebra be coming from a strict Lie 3-algebra as in example 13.1 and define a local 3-connection by the following maps:

$$\begin{aligned}
\mathrm{con}(\mathbf{a}^a) &= A^a \\
\mathrm{con}\big(\mathbf{b}^A\big) &= B^A \\
\mathrm{con}(\mathbf{c}^\alpha) &= C^\alpha
\end{aligned}$$

Recall that we required the connection to be a homogenizing morphism of dg-algebras and that $V_n^* = 0$ for $n > 3$ in $\mathfrak{p}_3(M)$.

This map is a chain map only if $[Q, \mathrm{con}_3] = 0$, which implies

$$\begin{aligned}
0 &= \mathbf{d}\mathrm{con}(\mathbf{a}^a) - \mathrm{con}\big(\mathbf{d}^{\mathfrak{g}}\mathbf{a}^a\big) \\
&= \mathbf{d}A^a + \frac{1}{2}C^a{}_{bc}A^b \wedge A^c + dt^a_A B^A \\
&= (F_A + dt(B))^a \,.
\end{aligned}$$

and

$$\begin{aligned}
0 &= \mathbf{d}\mathrm{con}\big(\mathbf{b}^A\big) - \mathrm{con}\big(\mathbf{d}^{\mathfrak{g}}\mathbf{b}^A\big) \\
&= \mathbf{d}B^A + (d\alpha_1)^A_{Ba}B^A A^a + (dt_2)^A_\alpha C^\alpha \\
&= (\mathbf{d}_A B + dt_2(C))^A \,.
\end{aligned}$$

(Note again that the relation at grade three becomes trivial due to the nature of $\mathfrak{p}_3$.)

These are precisely the consistency conditions known in the integral formalism from prop 12.1 (p. 314).

References

[1] F. Ribeiro, Prevent mistypings to the string class, 2005. http://blog.phpx.com/post/3/265.

[2] A. Connes, Noncommutative Geometry. Academic Press, 1994.

[3] E. Witten, Supersymmetry and Morse theory, *J. Diff Geom.* **17** (1982) 661.

[4] E. Witten, Global anomalies in string theory, in *Symposium on Anomalies, Geometry, Topology* (W. Bardeen and A. White, eds.), pp. 61–99. World Scientific, 1985.

[5] J. Fröhlich, O. Grandjean, and A. Recknagel, Supersymmetric quantum theory and (non-commutative) differential geometry, *Commun. Math. Phys.* **193** *(1998) 527-594*. `hep-th/9612205`.

[6] J. Fröhlich, O. Grandjean, and A. Recknagel, Supersymmetric quantum theory, non-commutative geometry, and gravitation, `hep-th/9706132` (1997).

[7] F. Lizzi and R. Szabo, Noncommutative geometry and spacetime gauge symmetries of string theory, *Chaos, Solitons, Fractals* **10** (1999) 445–458. `hep-th/9712206`.

[8] F. Lizzi and R. Szabo, Duality symmetries and noncommutative geometry of string spacetimes, *Commun. Math. Phys.* **197** *(1998) 667-712*. `hep-th/9707202`.

[9] F. Lizzi, Noncommutative geometry, strings and duality, *Int. J. Mod. Phys.* **B 14** *(2000) 2383-2396*. `hep-th/9906122`.

[10] A. Chamseddine, An effective superstring spectral action, *Phys. Rev.* **D 56** *(1997) 3555-3567*. `hep-th/9705153`.

[11] A. Chamseddine, The spectral action priciple in noncommutative geometry and the superstring, `hep-th/9701096` (1997).

[12] M. Douglas and C. Hull, D-branes and the noncommutative torus, *J. High Energy Phys.* **02** *(1998) 008* (1998). `hep-th/9711165`.

[13] N. Seiberg and E. Witten, String theory and noncommutative geometry, *J. High Energy Phys.* **9909** *(1999) 032*. `hep-th/9908142`.

[14] T. Banks, W. Fischler, S. Shenker, and L. Susskind, M theory as a matrix model: A conjecture, *Phys. Rev.* **D 55** *(1997) 5112-5128*. `hep-th/9610043`.

[15] A. Connes, M. Douglas, and A. Schwarz, Noncommutative geometry and matrix theory: Compactification on tori, *J. High Energy Phys.* **9802** *(1998) 003*. `hep-th/9711162`.

[16] T. Banks, TASI lectures on matrix theory, `hep-th/9911068` (1999).

[17] E. Witten, Non-commutative geometry and string field theory, *Nucl. Phys.* **B 268** *(1986) 253-294* (1986).

[18] E. Witten, The index of the Dirac operator in loop space, in *Elliptic Curves and Modular Forms in Algebraic Topology* (P. Landweber, ed.), vol. 1326 of *Lecture Notes in Mathematics*, pp. 161–181. Springer, 1988.

[19] N. Baas, B. Dundas, and J. Rognes, Two-vector bundles and forms of elliptic cohomology, in *Topology, Geometry and Quantum Field Theory* (U. Tillmann, ed.), Cambridge U. Press, 2004. `math.AT/0306027`.

[20] S. Stolz and P. Teichner, What is an elliptic object?, in *Topology, Geometry and Quantum Field Theory* (U. Tillmann, ed.), Cambridge U. Press, 2004.

[21] J. Baez, Higher Yang-Mills theory, hep-th/0206130 (2002).

[22] D. Freed and E. Witten, Anomalies in string theory with D-branes, hep-th/9907189 (2000).

[23] P. Aschieri and B. Jurčo, Gerbes, M5-brane anomalies and E_8 gauge theory, *J. High Energy Phys.* **10** *(2004) 068*. hep-th/0409200.

[24] R. Attal, Combinatorial stacks and the four-colour theorem, math.CO/0501231 (2005).

[25] J. Baez and J. Dolan, Categorification, in *Higher Category Theory* (E. Getzler and M. Kapranov, eds.), Contemp. Math. 230. Am. Math. Soc., 1998. math.QA/9802029.

[26] G. Segal, The definition of conformal field theory, in *Topology, Geometry and Quantum Field Theory* (U. Tillmann, ed.), Cambridge U. Press, 2004.

[27] U. Schreiber, On deformations of 2d SCFTs, *J. High Energy Phys.* **06** *(2004) 058*. hep-th/0401175.

[28] U. Schreiber, DDF and Pohlmeyer invariants of (super)string, *J. High Energy Phys.* **05** *(2004) 027*. hep-th/0403260.

[29] U. Schreiber, Pohlmeyer invariants are expressible in terms of DDF invariants, math-ph/0404007 (2004).

[30] U. Schreiber, Nonabelian 2-forms and loop space connections from 2d SCFT deformations, hep-th/0407122 (2004).

[31] J. Baez and U. Schreiber, Higher gauge theory: 2-connections on 2-bundles, hep-th/0412325*, revised version at http://math.ucr.edu/home/baez/2conn.pdf, to appear in* proceedings of "Categories in Algebra, Geometry and Mathematical Physics", conference in honor of Ross Street's 60th birthday (2005).

[32] J. Baez, A. Crans, U. Schreiber, and D. Stevenson, From loop groups to 2-groups, math.QA/0504123 (2005).

[33] U. Schreiber, Nonabelian deligne hypercohomology, *(in preparation) http://www-stud.uni-essen.de/˜sb0264/nad.pdf* (2005).

[34] U. Schreiber, Global surface holonomy, *(in preparation)* (2005).

[35] U. Schreiber, Super-Pohlmeyer invariants and boundary states for nonabelian gauge fields, *J. High Energy Phys.* **10** *(2004) 035*. hep-th/0408161.

[36] T. Bartels, Categorified gauge theory: 2-bundles, math.CT/0410328 (2004).

[37] T. Strobl, Algebroid Yang-Mills theories, *Phys. Rev. Lett.* **93** *(2004) 211601*. hep-th/0406215.

[38] The String Coffee Table, 2004. http://golem.ph.utexas.edu/string/index.shtml.

[39] E. Forgy and U. Schreiber, Discrete differential geometry on causal graphs, *math-ph/0407005* (2004).

[40] J. Baez and A. Crans, Higher-dimensional algebra VI: Lie 2-algebras, *Theory and Applications of Categories* **12** (2004) 492. math.QA/0307263.

[41] J. Baez and A. Lauda, Higher-dimensional algebra V: 2-groups, *Theory and Applications of Categories* **12** (2004) 423. math.QA/0307200.

[42] O. Alvarez, Topological quantization and cohomology, *Commun. Math. Phys.* **100** *(1985) 279* (1985).

[43] K. Gawedzki, Topological actions in two-dimensional quantum field theories, in *Nonperturbative quantum field theory* (G. 't Hooft et al., ed.). Plenum Press, New York, 1988.

[44] A. Kapustin, D-branes in topologically nontrivial B field, *Adv. Theor. Math. Phys.* **4** *(2000) 127-154*. `hep-th/9909089`.

[45] K. Gawedzki and N. Reis, WZW branes and gerbes, `hep-th/0205233` (2002).

[46] A. Carey, S. Johnson, and M. Murray, Holonomy on D-branes, `hep-th/0204199` (2004).

[47] D. Chatterjee, On Gerbs. PhD thesis, University of Cambridge, 1998.

[48] M. Mackaay and R. Picken, Holonomy and parallel transport for abelian gerbes, `math.DG/0007053` (2001).

[49] L. Breen and W. Messing, Differential geometry of gerbes, `math.AG/0106083` (2001).

[50] P. Aschieri, L. Cantini, and B. Jurčo, Nonabelian bundle gerbes, their differential geometry and gauge theory, *Commun. Math. Phys.* **254** *(2005) 367-400*. `hep-th/0312154`.

[51] O. Ganor, Six-dimensional tensionless strings in the large N limit, *Nucl. Phys.* **B 489** *(1997) 95-121*. `hep-th/9605201`.

[52] X. Bekaert, M. Henneaux, and A. Sevrin, Chiral forms and their deformations, *Commun. Math. Phys.* **224** *(2001) 683-703*. `hep-th/0004049`.

[53] C. Hofman, Nonabelian 2-forms, `hep-th/0207017` (2002).

[54] K. Lechner, Intersecting M2- and M5-branes, *Phys. Lett.* **B 589** *(2004) 147-154*. `hep-th/0402078`.

[55] P. Brax and J. Mourad, Open supermembranes coupled to M-theory fixe branes, *Phys. Lett.* **B 416** *(1998) 295-302*. `hep-th/9707246`.

[56] A. Strominger, Open p-branes, *Phys. Lett.* **B 383** *(1996) 44-47*. `hep-th/9512059`.

[57] E. Witten, Branes and the dynamics of qcd, *Nucl. Phys.* **B 507** *(1997) 658-690*. `hep-th/9706109`.

[58] P. Townsend, D-branes from M-branes, *Phys. Lett.* **B 373** *(1996) 68-75*. `hep-th/9512062`.

[59] C. Callan and J. Maldacena, Brane dynamics from the born-infeld action, *Nucl. Phys.* **B 513** *(1998) 198-212*. `hep-th/9708147`.

[60] G. Gibbons, Born-Infeld particles and Dirichlet p-branes, *Nucl. Phys.* **B 514** *(1998) 603-639*. `hep-th/9709027`.

[61] P. Howe, N. Lambert, and P. West, The self-dual string soliton, *Nucl. Phys.* **B 515** *(1998) 203-216*. `hep-th/9709014`.

[62] O. Aharony, A brief review of "little string theories", *Class. and Quant. Grav.* **17** *(2000) 929-938*. `hep-th/9911147`.

[63] A. Losev, G. Moore, and S. Shatashvili, M & m's, *Nucl. Phys.* **B 522** *(1998) 105-124* (1997). `hep-th/9707250`.

[64] J. Kalkkinen, Non-abelian gerbes from strings on a branched space-time, `hep-th/9910048` (1999).

[65] P. Brax and J. Mourad, Open supermembranes in eleven dimensions, *Phys. Lett.* **B 408** *(1997) 142-150*. `hep-th/9704165`.

[66] S. Kawamoto and N. Sasakura, Open membranes in a constant C-field background and non-commutative boundary strings, *J. High Energy Phys.* **07** *(2000) 014*. `hep-th/0005123`.

[67] E. Bergshoeff, D. Bergman, J. van der Schaar, and P. Sundell, The M5-brane and non-commutative loop space, *Class. and Quant. Grav.* **18** *(2001) 3265-3273*.

[68] A. Lahiri, Local symmetries of the non-Abelian two-form, *J. Phys.* **A 35** *(2002) 8779-8792*. `hep-th/010922`.

[69] A. Lahiri, Parallel transport on non-Abelian flux tubes, `hep-th/0312112`.

[70] N. Seiberg and E. Witten, Comments on string dynamics in six dimensions, *Nucl. Phys.* **B 471** *(1996) 121-134*. `hep-th/9603003`.

[71] E. Witten, Some comments on string dynamics, `hep-th/9507121` (1995).

[72] M. Duff and J. Lu, Black and super p-branes in diverse dimensions, *Nucl. Phys.* **B 416** *(1994) 301-334*. `hep-th/9306052`.

[73] E. Witten, Five-brane effective action in M-theory, *J. Geom. Phys.* **22** *(1997) 658-690*. `hep-th/9610234`.

[74] D. Berman and J. Harvey, The self-dual string and anomalies in the M5-brane, `hep-th/0408198` (2004).

[75] I. Klebanov and A. Tseytlin, Entropy of near-extremal black p-branes, *Nucl. Phys.* **B 475** *(1996) 164-178*. `hep-th/9604089`.

[76] S. Gubser and R. Klebanov, Absorption by branes and schwinger terms in the world volume theory, *Phys. Lett.* **B 413** *(1997) 41-48*. `hep-th/9708005`.

[77] S. Gubser, R. Klebanov, and A. Tseytlin, String theory and classical absorption by threebranes, *Nucl. Phys.* **B 499** *(1997) 217-240*. `hep-th/9703040`.

[78] M. Henningson and K. Skenderis, The holographic weyl anomaly, *J. High Energy Phys.* **07** *(1998) 023*. `hep-th/9806087`.

[79] F. Bastianelli, S. Frolov, and A. Tseytlin, Conformal anomaly of $(2, 0)$ tensor multiplet in six dimensions and AdS/CFT correspondence, *J. High Energy Phys.* **02** *(2000) 013*. `hep-th/0001041`.

[80] D. Berman, Aspects of M-5 brane world volume dynamics, *Phys. Lett.* **B 572** *(2003) 101-107* (2003). `hep-th/0307040`.

[81] O. Aharony, A. Hanany, and B. Kol, Webs of (p, q) 5-branes, five dimensional field theories and grid diagrams, *J. High Energy Phys.* **01** *(1998) 002*. `hep-th/9710116`.

[82] B. Kol and J. Rahmfeld, BPS spectrum of 5 dimensional field theories, (p, q) webs and curve counting, *J. High Energy Phys.* **08** *(1998) 006*. `hep-th/9801067`.

[83] D. Berenstein and R. Leigh, String junctions and bound states of intersecting branes, *Phys. Rev.* **D 60** *(1999) 026005*. `hep-th/9812142`.

[84] M. Fukuma, S. Hosono, and H. Kawai, Lattice topological field theory in two dimensions, *Commun. Math. Phys.* **161** *(1992) 157-176.* hep-th/9212154.

[85] E. Akhmedov, Towards the theory of non-abelian tensor fields I, hep-th/0503234 (2005).

[86] T. Killingback, World-sheet anomalies and loop geometry, *Nucl. Phys.* **B 288** *(1987) 578* (1987).

[87] M. Murray and D. Stevenson, Higgs fields, bundle gerbes and string structures, *Commun. Math. Phys.* **243** *(2003) 541-555.* math.DG/0106179.

[88] E. Witten, Elliptic genera and quantum field theory, *Commun. Math. Phys.* **109** *(1987) 525.*

[89] S. MacLane, Categories for the working mathematician. Springer, 1971.

[90] J. Baez and A. Lauda, A history of n-categorical physics, *(in preparation http://math.ucr.edu/home/baez/qg-winter2005/history.pdf)* (2005).

[91] H. Pfeiffer, Higher gauge theory and a non-Abelian generalization of 2-form electromagnetism, hep-th/0304074. *Ann. Phys. (NY)* **308** (2003) 447.

[92] F. Girelli and H. Pfeiffer, Higher gauge theory - differential versus integral formulation, *J. Math. Phys.* **45** *(2004) 3949-3971.* hep-th/0309173.

[93] W. Siegel, Introduction to string field theory, hep-th/0107049 (2001).

[94] K. Ohmori, A review on tachyon condensation in open string field theories, hep-th/0102085 (2001).

[95] N. Berkovits, Review of open superstring field theory, hep-th/0105230.

[96] W. Taylor and B. Zwiebach, D-branes, tachyons and string field theory, hep-th/0311017 (2003).

[97] J. Baez, Higher-dimensional algebra II: 2-Hilbert spaces, *Adv. Math* **127** (1997) 125–189.

[98] P. Aspinwall, D-branes on Calabi-Yau manifolds, hep-th/0403166 (2004).

[99] C. Lazaroiu, D-brane categories, *Int. J. Mod. Phys.* **A 18** *(2003) 5299-5335.* hep-th/0305095.

[100] M. Kapranov and V. Voevodsky, 2-categories and Zamolodchikov tetrahedra equations, in *Algebraic groups and their generalization* (W. Haboush, ed.), 1994.

[101] T. Bridgeland, Stability conditions on triangulated categories, math.AG/0212237 (2003).

[102] A. Gorodentscev, S. Kuleshov, and A. Rudakov, Stability data and t-structures on a triangulated category, math.AG/0312442.

[103] A. Yekutieli, Dualizing complexes, Morita equivalence and the derived Picard group of a ring, *J. London Math. Soc.* **60** (1999) 723–746.

[104] J. Miyachi and A. Yekutieli, The derived Picard group and representations of quivers, in *Proceedings of the 32nd Symposium on Ring Theory and Representation Theory* (J. Miyachi, ed.), 2000.

[105] J. Miyachi and A. Yekutieli, Derived Picard groups of finite dimensional hereditary algebras, *Composito Mathematica* **129** (2001) 341–368.

[106] A. Yekutieli, The derived Picard group is a locally algebraic group, *Algebras and representation theory* **7** (2004) 53–57.

[107] N. Arkani-Hamed, A. Cohen, D. Kaplan, A. Karch, and L. Motl, Deconstructing (2,0) and little string theories, *J. High Energy Phys.* **01** *(2003) 083*. `hep-th/0110146`.

[108] D. Berenstein and M. Douglas, Seiberg duality for quiver gauge theories, `hep-th/0207027` (2002).

[109] B. Feng, A. Hanany, Y. He, and A. Uranga, Toric duality as Seiberg duality and brane diamonds, `hep-th/0109063` (2002).

[110] S. Franco and A. Hanany, On the fate of tachyonic quivers, `hep-th/0408016` (2004).

[111] A. Polyakov, Gauge Fields and Strings. Harwood Academic Publisher, 1987.

[112] T. Hollowood and T. Kingaby, A comment on the χ_y genus and supersymmetric quantum mechanics, *Phys. Lett.* **B 566** *(2003) 258-262*. `hep-th/0303018`.

[113] U. Schreiber, Covariant Hamiltonian evolution in supersymmetric quantum systems, `hep-th/0311064` (2003).

[114] J. Azcarraga, J. Izquierdo, M. Picon, and O. Varela, Extensions, expansions, Lie algebra cohomology and enlarged superspaces, `hep-th/0401033` (2004).

[115] I. Giannakis, Superconformal deformations and space-time symmetries, `hep-th/9902194` (1999).

[116] B. Ovrut and S. K. Rama, Deformation of superconformal field theories and target-space symmetries, *Phys. Rev.* **D 44** *(1992) 550*.

[117] M. Evans and B. Ovrut, Deformations of conformal field theories and symmetries of the string, *Phys. Rev.* **D 41** *(1990) 3149*.

[118] M. Evans and B. Ovrut, Spontaneously broken inter mass level symmetries in string theory, *Phys. Lett.* **B 231** *(1989) 80*.

[119] J. Freericks and M. Halpern, Conformal deformation by the currents of affine g, *Ann. Phys. (NY)* **188** *(1988) 258* (1988).

[120] I. Giannakis, Strings in nontrivial gravitino and Ramond-Ramond backgrounds, `hep-th/0205219` (2002).

[121] M. Kato, BRST operators and similarity transformations, `hep-th/9512201` (1995).

[122] S. Förste and D. Roggenkamp, Current-current deformations of conformal field theories, and WZW models, *J. High Energy Phys.* **05** *(2003) 071*. `hep-th/0304234`.

[123] J. Fröhlich and K. Gawedzki, Conformal field theory and geometry of strings, `hep-th/9310187` (1993).

[124] M. Evans and I. Giannakis, T-duality in arbitrary string backgrounds, *Nucl. Phys.* **B 472** *(1996) 139-162*. `hep-th/9511061`.

[125] T. Wurzbacher, Symplectic geometry of the loop space of a Riemannian manifold, *J. Geom. Phys.* **16** *(1995) 345-384*.

[126] T. Wurzbacher, The Dirac-Ramond operator on loops in flat space, 2001.

[127] A. Arai, Dirac operators in boson-fermion Fock spaces and supersymmetric quantum field theory, *J. Geom. Phys.* **11** *(1993) 465*.

[128] U. Schreiber, Supersymmetric homogeneous quantum cosmology, *http://www-stud.uni-essen.de/˜sb0264/sqm.html* (2001).

[129] J. Polchinski, String Theory. Cambridge University Press, 1998.

[130] K. Bardakci and M. Halpern, New dual quark models, *Phys. Rev.* **D 3** *(1971) 2493* (1971).

[131] A. Chodos and C. Thorn, Making the massless string massive, *Nucl. Phys.* **B 72** *(1974) 509* (2002).

[132] M. Evans and I. Giannakis, Deformations, symmetries and topological degrees of freedom of the string, `hep-th/9109055` (1991).

[133] J. Bagger and I. Giannakis, Spacetime supersymmetry in a nontricorollary: curvature of standard path space 1-form al NS-NS superstring background, *Phys. Rev.* **D 65** *(2002) 046002*. `hep-th/0107260`.

[134] I. Bandos and B. Julia, Superfield T-duality rules in ten dimensions with one isometry, `hep-th/0312299` (2003).

[135] A. Giveon, M. Porrati, and E. Rabinovici, Target space duality in string theory, *Phys. Rept.* **244** *(1994) 77-202*. `hep-th/9401139`.

[136] M. Duff, Strong/weak coupling duality from the dual string, *Nucl. Phys.* **B 442** *(1995) 47-63*. `hep-th/9501030`.

[137] E. Witten, Bound states of strings and p-branes, *Nucl. Phys.* **B 460** *(1996) 335-350.* `hep-th/9510135`.

[138] Y. Igarishi, K. Itoh, K. Kamimura, and R. Kuriki, Canonical equivalence between super D-string and type IIB superstring, *J. High Energy Phys.* **03** *(1998) 002.*

[139] T. Thiemann, The LQG-string: Loop quantum gravity quantization of String Theory I. Flat target space, `hep-th/0401172`.

[140] K. Pohlmeyer and M. Trunk, The invariant charges of the Nambu-Goto theory: Quantization of non-additive composition laws, *Int. J. Mod. Phys.* **D 19** *(2004) 115-148.* `hep-th/0206061`.

[141] C. Meusburger and K. H. Rehren, Algebraic quantization of the closed bosonic string, *Commun. Math. Phys.* **237** *(2003) 69-85* (2002). `math-ph/0202041`.

[142] K. Pohlmeyer, The Nambu-Goto theory of closed bosonic strings moving in 1+3-dimensional Minkowski space: The quantum algebra of observables, *Ann. Phys. (NY)* **8** *(1999) 19-50.* `hep-th/9805057`.

[143] K. Pohlmeyer and K.-H. Rehren, The invariant charges of the Nambu-Goto theory: Their geometric origin and their completeness, *Commun. Math. Phys.* **114** *(1988) 177.*

[144] V. Borodulin and A. Isaev, On the infinite set of integrals of motion for a closed supersymmetric string, *Phys. Lett.* **B 117** *(1982) 69.*

[145] A. Isaev, Integrals of the motion of a closed relativistic string, *Theor. Math. Phys.* **54** *(1983) 134.*

[146] D. Bahns, The invariant charges of the Nambu-Goto string and canonical quantization, `hep-th/0403108` (2004).

[147] E. D. Giudice, P. D. Vecchia, and A. Fubini, General properties of the dual resonance model, *Ann. Phys. (NY)* **70** *(1972) 378-398*.

[148] M. Green, J. Schwarz, and E. Witten, Superstring Theory. Cambridge University Press, 1987.

[149] H. Aoki, S. Iso, H. Kawai, Y. Kitazawa, T. Tada, and A. Tsuchiya, IIB Matrix Model, *Prog. Theor. Phys. Suppl.* **134** *(1999) 47-83*. hep-th/9908038.

[150] K.-H. Rehren, 2004. (private communication).

[151] S. Carlip, Quantum gravity: A progress report, *Rep. Prog. Phys.* **64** (2001) 885. hep-th/0108040.

[152] T. Maeda, T. Nakatsu, and T. Oonishi, Non-linear field equation from boundary state formalism, hep-th/0312260 (2004).

[153] K. Murakani and T. Nakatsu, Open wilson lines as states of closed string, *Prog. Theor. Phys.* **110** *(2003) 285*. hep-th/0211232.

[154] K. Hashimoto, Generalized supersymmetric boundary state, *J. High Energy Phys.* **04** *(2000) 023*.

[155] K. Hashimoto, Correction to D-brane action and generalized boundary state, *Phys. Rev.* **D 61** *(2000) 106002*. hep-th/9909027.

[156] U. Schreiber, Loop space methods in string theory, *http://www-stud.uni-essen.de/~sb0264/looptalk.pdf* (2004).

[157] O. Andreev and A. Tseytlin, Representation for the open superstring effective action: Cancellation of Möbius infinities and derivative corrections to Born-Infeld lagrangian, *Nucl. Phys.* **B 311** *(1988) 205* (1988).

[158] E. Getzler, D. Jones, and S. Petrack, Differential forms on loop spaces and the cyclic bar complex, *Topology* **30** (1991), no. 3 339.

[159] Y. Zunger, Constructing exotic D-branes with infinite matrices in type IIA string theory, hep-th/0210175 (2002).

[160] O. Alvarez, L. Ferreira, and J. Sánchez Guillén, A new approach to integrable theories in any dimension, *Nucl. Phys.* **B 529** *(1998) 689* (1998). hep-th/9710147.

[161] I. Chepelev, Non-abelian Wilson surfaces, *J. High Energy Phys.* **02** *(2002) 013*.

[162] A. Recknagel and V. Schomerus, Boundary deformation thoery and moduli spaces of D-branes, *Nucl. Phys.* **B 545** *(1999) 233-282*. hep-th/9811237.

[163] V. Schomerus, Lectures on branes in curved backgrounds, *Class. and Quant. Grav.* **19** *(2002) 5781-5847*. hep-th/0209241.

[164] M. Baumgartl, On boundary states in superstring theory, *http://www.theorie.physik.uni-muenchen.de/~marco/da-download.php* (2003).

[165] C. Ehresmann, Introduction to the theory of structured categories, 1996.

[166] K. Chen, Iterated integrals, fundamental groups and covering spaces, *Trans. Amer. Math. Soc.* **206** (1975) 83–98.

[167] R. Brown and C. Spencer, $\mathcal{G}$-groupoids, crossed modules, and the classifying space of a topological group, *Proc. Kon. Akad. v. Wet.* **79** (1976) 296–302.

[168] R. Brown, Groupoids and crossed objects in algebraic topology, *Available online at http://www.math.rutgers.edu/hha/volumes/1999/volume1-1.htm*.

[169] L. Crane and M. Sheppeard, 2-categorical poincaré representations and state sum applications, `math.QA/0306440`.

[170] L. Crane and D. Yetter, Measurable categories and 2-groups, `math.QA/0305176`.

[171] F. Thomson, Introducing quaternionic gerbes, `math.DG/0009201` (2000).

[172] A. Pressley and G. Segal, Loop Groups. Oxford U. Press, 1986.

[173] A. Grothendieck, Pursuing stacks, *Available online at http://www.math.jussieu.fr/∼leila/mathtexts.php*.

[174] J. Giraud, Cohomologie non abelienne. Springer Verlag, Berlin-New York, 1971.

[175] I. Moerdijk, Introduction to the language of stacks and gerbes, `math.AT/0212266` (2002).

[176] M. Murray, Bundle gerbes, *J. London Math. Soc.* **54** (1996) 403–416. `math.DG/9407015`.

[177] D. Stevenson, The geometry of bundle gerbes, `math.DG/0004117` (2000).

[178] J.-L. Brylinski, Loop Spaces, Characteristic Classes and Geometric Quantization. Birkhauser, 1993.

[179] J. Brylinski and D. McLaughlin, The geometry of degree 4 characteristic classes and of line bundles on loop spaces, *Duke Math. J.* **75** (2004) 603–638.

[180] A. Carey, S. Johnson, M. Murray, and D. Stevenson, Bundle gerbes for chern-simons and wess-zumino-witten theories, `math.DG/0410013` (2004).

[181] J. Brylinski, Differentiable cohomology of gauge groups, `math.DG/0011069`.

[182] S. Chern and J. Simons, Characteristic forms and geometric invariants, *Ann. Math.* **99** (1974) 48–69.

[183] J. Cheeger and J. Simons, Differential characters and geometric invariants, in *Geometry and Topology* (J. Alexander and J. Harer, eds.), Lecture Notes in Mathematics **1167**, pp. 50–80. 1985.

[184] M. Schlessinger and J. Stasheff, The Lie algebra structure of tangent cohomology and deformation theory, *Jour. Pure App. Alg.* **38** (1985) 313–322.

[185] T. Lada and M. Markl, Strongly homotopy Lie algebras, `hep-th/9406095` (1994).

[186] J. Milnor, Remarks on infinite dimensional lie groups, in *Relativity, Groups and Topology II, (Les Houches, 1983)*. 1984.

[187] R. Hamilton, The inverse function theorem, *Bull. Amer. Math. Soc.* **7** (1982) 65–222.

[188] J. Mickelsson, Kac–Moody groups, topology of the Dirac determinant bundle, *Commun. Math. Phys.* **110** *(1987) 173-183* (1987).

[189] G. Segal, Classifying spaces and spectral sequences, *Publ. Math. IHES* **34** (1968) 105–112.

[190] A. Caetano and R. Picken, An axiomatic definition of holonomy, *Int. J. Math.* **5** (1993), no. 6 835–848.

[191] R. Karp, F. Mansouri, and J. Rno, Product integral formalism and non-abelian Stokes theorem, *J. Math. Phys.* **40** *(1999) 6033.* `hep-th/9910173`.

[192] O. Alvarez, 2004. (private communication).

[193] J. Baez, Torsors made easy, 2004. http://math.ucr.edu/home/baez/torsors.html.

[194] T. Lada and J. Stasheff, Introduction to sh Lie algebras for physicists, *Int. J. Theor. Phys.* **32** *(1993) 1087-1104*. `hep-th/9209099`.

[195] G. Barnich, R. Fulp, T. Lada, and J. Stasheff, The sh Lie structure of Poisson brackets in field theory, `hep-th/9702176` (1997).

[196] M. Bojowald, A. Kotov, and T. Strobl, Lie algebroid morphism, Poisson sigma models, and off-shell closed gauge symmetries, *J. Geom. Phys.* **54** *(2005) 400-426*. `math.DG/0406445`.

[197] A. Kotov, P. Schaller, and T. Strobl, Dirac sigma models, `hep-th/0411112` (2004).

[198] K. Mackenzie, Lie algebroids and Lie pseudoalgebras, *Bull. London Math. Soc.* **27** (1995) 97–147.

[199] M. Grützmann and T. Strobl, General Yang-Mills type gauge theories for p-form gauge fields: a physicist approach, *(in preparation)* (2005).